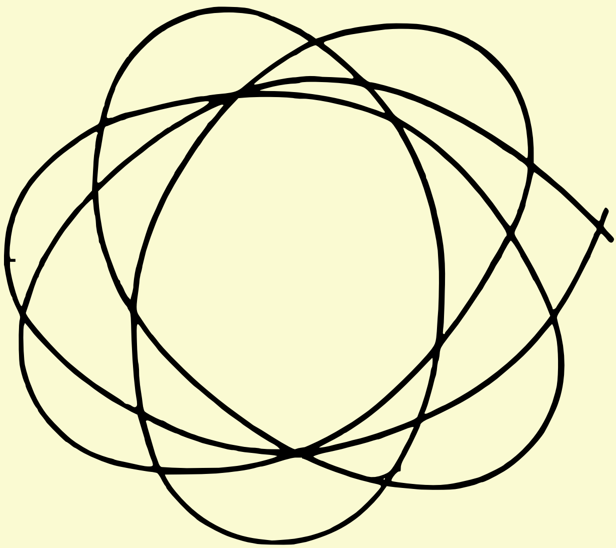
V. M. STARZHINSKII

APPLIED METHODS IN THE THEORY OF NONLINEAR OSCILLATIONS



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В. М. СТАРЖИНСКИЙ

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APPLIED METHODS IN THE THEORY OF NONLINEAR OSCILLATIONS

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PREFACE

Progress in the theory of nonlinear oscillations during the last decades was based chiefly on classical methods developed in the late 19th and early 20th centuries. This is illustrated by developments in the method of small parameter in monographs by Andronov, Vitt, and Khaikin [5], as well as Bulgakov [31], and Malkin [111a, b], developments in the method of averaging based on the Van der Pol method (Bogolyubov and Mitropolskii [22, 127d], Volosov and Morgunov [204]), by a new perturbation theory (Arnold [215]) based on classical perturbation methods, and by the Kamenkov V-function method [83, vol. II] based on fundamental results obtained by Lyapunov and Chetaev.

At the same time, new methods penetrated the theory of nonlinear oscillations: asymptotic methods developed by Bogolyubov, Krylov, and Mitropolskii [22, 102, 24, 127c], analytic functions methods introduced by Krasnoselskii [97a, b, 297a-d] and his school [98, 99], the method of point transformations developed by Andronov and Vitt [4, 5], and Neimark [137a, b], the stroboscopic method (Minorsky [125a, b, c]), the Gantmacher-Krein oscillation method [62], and the method of determining conditionally periodic motions introduced by Kolmogorov and Arnold [215, 286]. The idea of a new method is relative, of course, if we recall that Euler, Lagrange, and Laplace used averaging long before Van der Pol. This remark, however, is meant for future investigators.

Part One of the book is devoted to the combination of the Lyapunov, Poincaré, and averaging methods as applied to the analysis of oscillations in Lyapunov and nearly Lyapunov systems. A topic of interest is the investigation of oscillatory systems represented by analytic autonomous differential equations having no small parameters. The Lyapunov method of finding periodic solutions is known for the case of a conservative system (Lyapunov systems). The periodic solution obtained by means of the Lyapunov method depends, however, only on two constants of integration. Therefore it cannot in principle be a general solution for systems with more than two degrees of freedom; moreover, cases are known when the Lyapunov method fails. Chapter I, Section 1 discusses a transfor-

mation, outlined by Lyapunov, of an initial system whereby the system's order is lowered by two, a parameter equal to the square root of reduced energy is introduced, and the system becomes nonautonomous. If this parameter is sufficiently small, the methods of small parameter can be applied to the transformed quasilinear nonautonomous system.

This modification of the method proved effective in a number of problems, in particular, the problem of energy transfer. The first step is to determine the initial periodic mode and find its instability regions in the space of the system's parameters using the theory of parametric resonance [111b, 80]. The second step consists in determining the periodic modes that appear at critical values of the parameters and are, of course, distinct from the initial mode. This step uses the above-mentioned transformations and the Poincaré method of finding periodic solutions for nonautonomous systems. Other methods of small parameter can also be used with the transformed system, for instance, the method of averaging; in this case it becomes possible to carry out the third step of analysis, namely, investigation of the transient process, often referred to as energy transfer. All three steps are illustrated in Chapter III for the spring-loaded pendulum, pendulum subject to elastic suspension, and betatron oscillations of particles in cyclic accelerators with weak focusing. Note that the energy transfer problem is based on the general theory of oscillatory chains presented in Chapter II.

The next point is the application of perturbation theory (Chapter IV, Section 1). We assume that an unperturbed Lyapunov-type nonlinear autonomous system of order $2k + 2$ is perturbed by an analytic, and sufficiently small in norm, damping. A transformation of the perturbed system is carried out in which the unperturbed system is converted into a quasilinear nonautonomous system of order $2k$. Its solution is assumed known for a sufficiently small (compared to unity) square root of the initial value of the reduced energy of the system. In order to find the first- and higher-order corrections of the corresponding solution (i.e. with the same initial conditions) of the perturbed system, we must write a complete set of variational equations, that is, a sequence of nonhomogeneous systems of linear differential equations of order $2k + 1$ with variable coefficients. The complete system is given in operator form for the general finite-dimensional case of analytic perturbation theory. According to Poincaré, the integration of the complete system is reduced to quadratures provided a general integral of the unperturbed system is known.

The last section of the first part of the book treats oscillations in Lyapunov-type systems. We present here some of the results obtained by Nustrov [336a, b]; the table of contents gives a fair idea of the subjects discussed.

The second part of the book is also based on the results achieved in one of the classical methods developed in the years spanning the late 19th and early 20th centuries, the theory of normal forms (Poincaré, Lyapunov, Dulac, Siegel, Moser, Arnold, Pliss, and others).

Brjuno [238a-v] obtained general results in the theory of normal forms of nonlinear analytic autonomous systems of ordinary differential equations. The method was first introduced by Poincaré [149a]. The theory is applied in the second part of the book to analyze oscillations described by such equations.

Chapter V gives the elements of the theory of normal forms required to understand the material.

In Chapter VI we single out the class of problems in which the normal form has the simplest form as given by the Poincaré theorem and in which the general solution of the Cauchy problem can be obtained at each step of the approximation efficiently. This class covers damped oscillatory systems (asymptotically stable in linear approximation) with analytic nonlinearities of the general type. The results are illustrated by two examples of mechanical systems with one and two degrees of freedom.

In the next chapter we consider third-order systems with two pure imaginary and one negative (Chapter VII, Section 1) or vanishing (Chapter VII, Section 2) eigenvalues of the linear part. Chapter VII, Section 1 concludes with an investigation of oscillations in electromechanical systems with "one and a half" degrees of freedom.

Finally, normal forms and resonances are studied in analytic fourth-order (Chapter VIII, Section 1) and sixth-order (Chapter VIII, Section 4) autonomous systems with two and three pairs, respectively, of various pure imaginary (in general, nonconservative) eigenvalues of the matrix of the linear part. The Cauchy problem is solved in the general case with quadratic terms included. The results derived from the Molchanov and Bibikov-Pliss stability criteria are discussed for third-power normal forms. Two methods are suggested for constructing the Lyapunov function for the case of conservative systems: a direct method, and one based on Chetaev's linear combination of integrals obtained by means of third-power normal forms. The results are applied to the Ishlinskii problem concerning the motion of the gyroscopic frame of a sensor element of a gyroscopic compass (Chapter VIII, Section 2).

In the first approximation, the two parts of the book are independent.

The book is aimed at engineers with a strong mathematical background, scientists working in mechanics and applied mathematics, and undergraduate and postgraduate students of Applied Physics and Physics and Mathematics departments.

The book is based on a course of lectures presented by the author to engineering students at the Mechanics and Mathematics Department of Moscow University in 1956-1976.

If the author has been successful in giving the reader an insight into the theory of oscillations and stability, he owes this primarily to the late Boris V. Bulgakov and Nikolai G. Chetaev.

The formulas within each subsection of the text are numbered without citing the section number. If a formula of another section is cited, the number of this section is added to the formula number; if the formula cited belongs to a different chapter, the number of this chapter is written in front of the section number and is separated by a comma. The same rule holds when sections and subsections are cited.

V. Starzhinskii

CONTENTS

PART ONE

OSCILLATIONS IN LYAPUNOV SYSTEMS

* * *

CHAPTER I

INTRODUCTION

§ 1. Transformation of Lyapunov Systems

The order of Lyapunov systems ([108a], §§ 33-45; [111a], Ch. IV; [111b], Ch. VII) can be reduced by two by using the energy integral and choosing, following Lyapunov, the polar angle in the plane of critical variables as the independent variable. The transformed system [371e-g, j, n, s, t] is nonautonomous and includes a parameter equal to the square root of the reduced constant energy. If this parameter is sufficiently small compared to unity, the Poincaré method ([188a], vol. I, Ch. III) of determining periodic solutions of nonautonomous systems (see Section 2 and Chapter III of this book) can be applied to the transformed system. The application of the Poincaré method is of special interest when the Lyapunov method ([108a], §§ 34-45; [111a], §§ 26-29; [111b], Ch. VII, §§ 1-4) cannot be applied to finding periodic solutions of the initial system.

In general, however, other methods of small parameter, for example, the method of averaging [150, 22, 127d, 204, 66a], can be applied to a transformed system. Since this allows us to not only determine periodic solutions but examine a broader range of problems, such as transient processes and so on, the usefulness of transforming Lyapunov systems becomes apparent. This aspect of the problem is discussed in Chapter III.

1.1. General case [371 e-g, j, s]. We consider a system of Lyapunov differential equations

$$\frac{dx}{dt} = -\lambda y + X(x,\ y,\ x_1,\ \ldots,\ x_n),$$

$$\frac{dy}{dt} = \lambda x + Y(x,\ y,\ x_1,\ \ldots,\ x_n),$$

$$\frac{dx_s}{dt} = p_{s1}x_1 + \ldots + p_{sn}x_n + X_s(x,\ y,\ x_1,\ \ldots,\ x_n) \qquad (1.1)$$

$$(s = 1,\ \ldots,\ n).$$

Here λ, p_{sr} are real constants, and X, Y, X_1, . . ., X_n are real analytic functions of x, y, x_1, . . ., x_n whose expansions begin with terms of order not lower than two. Lyapunov proved the following **theorem** ([108a], § 42): *if*

(a) *the equation* $\det \| p_{sr} - \delta_{sr}\varkappa \| = 0$ *has no roots of the type* $m\lambda i$ ($m = 0, \pm 1, \pm 2, \ldots, i = \sqrt{-1}$), *and*

(b) *it is possible to find power series in an arbitrary constant c satisfying system* (1.1)

$$\begin{aligned} x &= cx^{(1)} + c^2x^{(2)} + \ldots, \\ y &= cy^{(1)} + c^2y^{(2)} + \ldots, \\ x_s &= cx_s^{(1)} + c^2x_s^{(2)} + \ldots \quad (s = 1, \ldots, n), \end{aligned} \tag{1.2}$$

where $x^{(k)}$, $y^{(k)}$, $x_1^{(k)}$, . . ., $x_n^{(k)}$ ($k = 1, 2, \ldots$) *are periodic functions of t with the same period, and* $x^{(k)}(t_0) = y^{(k)}(t_0) = 0$ *for* $k > 1$, *then the series found are absolutely convergent if c remains below a certain limit and for these values of c the series are a periodic solution of the initial system* (1.1).

Let us analyze the cases in which at least one of these conditions is violated and the Lyapunov theorem consequently does not hold. If condition (a) is violated, we have the "resonant" case discussed by Ryabov [355a]. Condition (b) is violated if the expansions of X, Y, X_1, . . ., X_n do not contain the terms x^ν and y^ν ($\nu = 2, 3, \ldots$). In the latter case each coefficient of each of series (1.2) will be identically zero at each step of the calculation.

In Chapter III we demonstrate, however, that even in these cases it is possible to find periodic solutions of system (1.1) provided they exist. With a view to the remarks made in the introduction to this section, we shall consider a transformation of a Lyapunov system not bound, in general, by conditions (a) and (b). We assume in what follows that system (1.1) possesses a first integral,* which is inevitably an analytic function of x, y, x_1, . . ., x_n ([108a], § 38; [111a], § 25; [111b], Ch. VII, § 1) of the type

$$H = x^2 + y^2 + W(x_1, \ldots, x_n) + S_3(x, y, x_1, \ldots, x_n) = \mu^2 \quad (\mu > 0), \tag{1.3}$$

where W is a quadratic form. The Lyapunov substitution

$$\begin{aligned} x &= \rho \cos \vartheta, \\ y &= \rho \sin \vartheta, \\ x_s &= \rho z_s \quad (s = 1, \ldots, n) \end{aligned} \tag{1.4}$$

* This is included in the definition of a Lyapunov system.

transforms system (1.1) and the first integral (1.3) to the form

$$\frac{d\rho}{dt}=\rho^2 R(\rho,\ \vartheta,\ \mathbf{z}),$$
$$\frac{d\vartheta}{dt}=\lambda+\rho\Theta(\rho,\ \vartheta,\ \mathbf{z}),$$
$$\frac{dz_s}{dt}=p_{s1}z_1+\ldots+p_{sn}z_n+\rho Z_s(\rho,\ \vartheta,\ \mathbf{z}) \qquad (s=1,\ \ldots,\ n), \tag{1.5}$$
$$\rho^2[1+W(\mathbf{z})+\rho S(\rho,\ \vartheta,\ \mathbf{z})]=\mu^2. \tag{1.6}$$

Here R, Θ, $Z_1, \ldots, Z_n$, and S are analytic functions of the variables ρ, $z_1, \ldots, z_n$ in some neighbourhood of zero values whose expansions in powers of ρ, in general, begin with zero-power terms; the coefficients of power series in ρ, $z_1, \ldots, z_n$ are periodic functions of ϑ with period 2π that are polynomials with respect to $\cos\vartheta$ and $\sin\vartheta$

$$\begin{aligned} R &= \rho^{-2}[X(\rho\cos\vartheta,\ \rho\sin\vartheta,\ \rho\mathbf{z})\cos\vartheta \\ &\qquad + Y(\rho\cos\vartheta,\ \rho\sin\vartheta,\ \rho\mathbf{z})\sin\vartheta], \\ \Theta &= \rho^{-2}[-X\sin\vartheta+Y\cos\vartheta], \\ Z_s &= \rho^{-2}X_s(\rho\cos\vartheta,\ \rho\sin\vartheta,\ \rho\mathbf{z})-z_sR(\rho,\ \vartheta,\ \mathbf{z}) \\ &\qquad (s=1,\ \ldots,\ n), \\ S &= \rho^{-3}S_3(\rho\cos\vartheta,\ \rho\sin\vartheta,\ \rho\mathbf{z}) \end{aligned} \tag{1.7}$$

and $\mathbf{z}$ is a vector with the components $z_1, \ldots, z_n$.

We assume now that $1+W>0$ in (1.6). This holds true for all values of $z_1, \ldots, z_n$ if W is a positive-definite quadratic form (this is true for the energy integral) and for sufficiently small values of $z_1, \ldots, z_n$ in the general case. We solve equation (1.6) with respect to ρ for one selected branch of the analytic function; specifically, we presume only the arithmetic value of the root:

$$\begin{aligned}\rho=(1+W)^{-1/2}\mu\Big\{1-\frac{1}{2}(1+W)^{-3/2}S(0,\ \vartheta,\ \mathbf{z})\mu \\ +\Big[\frac{5}{8}(1+W)^{-3}S^2(0,\ \vartheta,\ \mathbf{z})-\frac{1}{2}(1+W)^{-2}S'_\rho(0,\ \vartheta,\ \mathbf{z})\Big]\mu^2\Big\} \\ +O(\mu^4).\end{aligned} \tag{1.8}$$

Assuming μ to be sufficiently small, we introduce phase time ϑ, by dividing the last n equations (1.5) by the second

$$\frac{dz_s}{d\vartheta}=\frac{p_{s1}z_1+\ldots+p_{sn}z_n+\rho Z_s(\rho,\ \vartheta,\ \mathbf{z})}{\lambda+\rho\Theta(\rho,\ \vartheta,\ \mathbf{z})}$$
$$(s=1,\ \ldots,\ n).$$

Substitution of expansion (1.8) into the above system yields

$$\lambda \frac{dz_s}{d\vartheta} = p_{s1}z_1 + \ldots + p_{sn}z_n + [1+W(\mathbf{z})]^{-1/2}\Big[Z_s(0, \vartheta, \mathbf{z})$$
$$- \frac{1}{\lambda}(p_{s1}z_1 + \ldots + p_{sn}z_n)\,\Theta(0, \vartheta, \mathbf{z})\Big]\mu + [1+W(\mathbf{z})]^{-1}$$
$$\times \Big\{\frac{\partial Z_s(0, \vartheta, \mathbf{z})}{\partial\rho} - \frac{1}{2}[1+W(\mathbf{z})]^{-1} S(0, \vartheta, \mathbf{z})\, Z_s(0, \vartheta, \mathbf{z})$$
$$- \frac{1}{\lambda}\Theta(0, \vartheta, \mathbf{z})\, Z_s(0, \vartheta, \mathbf{z})$$
$$+ \frac{1}{\lambda}(p_{s1}z_1 + \ldots + p_{sn}z_n)\Big[\frac{1}{\lambda}\Theta^2(0, \vartheta, \mathbf{z}) - \frac{\partial\Theta(0, \vartheta, \mathbf{z})}{\partial\rho}$$
$$+ \frac{1}{2}(1+W)^{-1} S(0, \vartheta, \mathbf{z})\,\Theta(0, \vartheta, \mathbf{z})\Big]\Big\}\mu^2 + O(\mu^3) \tag{1.9}$$
$$(s = 1, \ldots, n).$$

System (1.9) was first derived by Malkin ([111a], § 26), who gave an explicit expression only for its constant part. Malkin's proof is restricted to the existence of a periodic solution in some neighbourhood of a vanishing generating solution. In this book (Chapter III) we wish to find periodic solutions in some neighbourhood of non-vanishing generating solutions as well as to analyze transient processes. The order of system (1.9) is by two units lower than that of the initial system (1.1), and the coefficients of its nonlinear part are periodic functions of ϑ with period 2π analytically dependent on $z_1, \ldots, z_n$ and on the small parameter μ.

1.2. Systems of second-order equations [371j, n, s, t]. We consider a class of Lyapunov systems represented by $k+1$ second-order equations

$$\frac{d^2u}{dt^2} + \lambda^2 u = U(u, \dot{u}, v_1, \ldots, v_k, \dot{v}_1, \ldots, \dot{v}_k),$$
$$\frac{d^2v_\varkappa}{dt^2} + a_{\varkappa 1}v_1 + \ldots + a_{\varkappa k}v_k = V_\varkappa(u, \dot{u}, v_1, \ldots, v_k, \dot{v}_1, \ldots, \dot{v}_k) \tag{2.1}$$
$$(\varkappa = 1, \ldots, k).$$

Here $\lambda > 0$; $a_{j\varkappa} = a_{\varkappa j}$ $(\varkappa, j = 1, \ldots, k)$ are real constants, and $U, V_1, \ldots, V_k$ are real analytic functions of $u, \dot{u}, v_1, \ldots, v_k, \dot{v}_1, \ldots, \dot{v}_k$ whose power expansions begin with terms of not lower than second order. We assume that system (2.1) allows a first integral, which inevitably is an analytic function of $u, \dot{u}, v_1, \ldots, v_k, \dot{v}_1, \ldots, \dot{v}_k$ ([108a], § 38; [111a], § 25) of the type

$$H = \dot{u}^2 + \lambda^2u^2 + W(v_1, \ldots, v_k, \dot{v}_1, \ldots, \dot{v}_k)$$
$$+ S_3(u, \dot{u}, v_1, \ldots, v_k, \dot{v}_1, \ldots, \dot{v}_k) = \mu^2 \quad (\mu > 0), \tag{2.2}$$

where W is a quadratic form and S_3 stands for all terms of not lower than third order. The Lyapunov substitution ([108a], § 33)

$$u=\frac{1}{\lambda}\rho\sin\vartheta, \qquad \dot{u}=\rho\cos\vartheta,$$

$$v_\varkappa=\rho z_\varkappa, \qquad \dot{v}_\varkappa=\rho z_{k+\varkappa} \quad (\varkappa=1, \ldots, k) \tag{2.3}$$

transforms system (2.1) and the first integral (2.2) to the form

$$\begin{aligned}
\frac{d\rho}{dt}&=\rho^2 R(\rho, \vartheta, \mathbf{z}),\\
\frac{d\vartheta}{dt}&=\lambda+\rho\Theta(\rho, \vartheta, \mathbf{z}),\\
\frac{dz_\varkappa}{dt}&=z_{k+\varkappa}+\rho Z_\varkappa(\rho, \vartheta, \mathbf{z}), \quad (\varkappa=1, \ldots, k),\\
\frac{dz_{k+\varkappa}}{dt}&=-a_{\varkappa 1}z_1-\ldots-a_{\varkappa k}z_k+\rho Z_{k+\varkappa}(\rho, \vartheta, \mathbf{z}),
\end{aligned} \tag{2.4}$$

$$\rho^2[1+W(\mathbf{z})+\rho S(\rho, \vartheta, \mathbf{z})]=\mu^2. \tag{2.5}$$

As in Subsection 1.1, here R, Θ, Z_1, . . ., Z_{2k}, and S are analytic functions of ρ, z_1, . . ., z_{2k} in some neighbourhood of zero values whose power expansions begin, in general, with zero-order terms; the coefficients of power series in the variables ρ, z_1, . . ., z_{2k} are periodic functions of ϑ with period 2π

$$\begin{aligned}
R &= \rho^{-2}U(\lambda^{-1}\rho\sin\vartheta, \rho\cos\vartheta, \rho\mathbf{z})\cos\vartheta,\\
\Theta &= -\rho^{-2}U(\lambda^{-1}\rho\sin\vartheta, \rho\cos\vartheta, \rho\mathbf{z})\sin\vartheta,\\
Z_\varkappa &= -z_\varkappa R(\rho, \vartheta, \mathbf{z}), \quad (\varkappa=1, \ldots, k),\\
Z_{k+\varkappa} &= \rho^{-2}V_\varkappa(\lambda^{-1}\rho\sin\vartheta, \rho\cos\vartheta, \rho\mathbf{z})-z_{k+\varkappa}R(\rho, \vartheta, \mathbf{z}),\\
S &= \rho^{-3}S_3(\lambda^{-1}\rho\sin\vartheta, \rho\cos\vartheta, \rho\mathbf{z})
\end{aligned} \tag{2.6}$$

and $\mathbf{z}$ is a vector with components z_1, . . ., z_{2k}. Assuming $1+W(\mathbf{z})>0$ in equation (2.5) (see Subsection 1.1), we solve (2.5) with respect to ρ

$$\rho=[1+W(\mathbf{z})]^{-1/2}\mu\left\{1-\frac{1}{2}[1+W(\mathbf{z})]^{-3/2}S(0, \vartheta, \mathbf{z})\mu\right\}+O(\mu^3). \tag{2.7}$$

We assume μ to be so small that the right-hand side of the second equation of system (2.4) is positive, that is,

$$\lambda+\rho(\vartheta, \mathbf{z}; \mu)\Theta(\rho(\vartheta, \mathbf{z}; \mu), \vartheta, \mathbf{z})>0,$$

and introduce phase time ϑ by dividing the last two groups of equations (2.4) by the second equation

$$\frac{dz_\varkappa}{d\vartheta}=\frac{z_{k+\varkappa}+\rho(\vartheta, \mathbf{z};\mu)Z_\varkappa(\rho(\vartheta, \mathbf{z};\mu), \vartheta, \mathbf{z})}{\lambda+\rho(\vartheta, \mathbf{z};\mu)\Theta(\rho(\vartheta, \mathbf{z};\mu), \vartheta, \mathbf{z})},$$

$$\frac{dz_{k+\varkappa}}{d\vartheta}=\frac{-a_{\varkappa 1}z_1-\ldots-a_{\varkappa k}z_k+\rho(\vartheta, \mathbf{z};\mu)Z_{k+\varkappa}(\rho(\vartheta, \mathbf{z};\mu), \vartheta, \mathbf{z})}{\lambda+\rho(\vartheta, \mathbf{z};\mu)\Theta(\rho(\vartheta, \mathbf{z};\mu), \vartheta, \mathbf{z})}$$

$$(\varkappa=1, \ldots, k).$$

The result of substituting expansion (2.7) into the above system is

$$\lambda\frac{dz_\varkappa}{d\vartheta}=z_{k+\varkappa}+\mu[1+W(\mathbf{z})]^{-1/2}[Z_\varkappa(0, \vartheta, \mathbf{z})-\lambda^{-1}z_{k+\varkappa}\Theta(0, \vartheta, \mathbf{z})]$$
$$+O(\mu^2), \quad (\varkappa=1, \ldots, k), \tag{2.8}$$
$$\lambda\frac{dz_{k+\varkappa}}{d\vartheta}=-a_{\varkappa 1}z_1-\ldots-a_{\varkappa k}z_k+\mu[1+W(\mathbf{z})]^{-1/2}$$
$$\times[Z_{k+\varkappa}(0, \vartheta, \mathbf{z})+\lambda^{-1}(a_{\varkappa 1}z_1+\ldots+a_{\varkappa k}z_k)\Theta(0, \vartheta, \mathbf{z})]+O(\mu^2).$$

Eliminating $z_{k+1}, \ldots, z_{2k}$ from this system, we arrive at

$$\lambda^2\frac{d^2z_\varkappa}{d\vartheta^2}+a_{\varkappa 1}z_1+\ldots+a_{\varkappa k}z_k=\mu[1+W(\mathbf{z})]^{-1/2}$$
$$\times\Big[Z_{k+\varkappa}+\lambda\frac{\partial Z_\varkappa}{\partial\vartheta}+2\lambda^{-1}(a_{\varkappa 1}z_1+\ldots+a_{\varkappa k}z_k)\Theta$$
$$-\lambda z'_\varkappa\frac{\partial\Theta}{\partial\vartheta}+\lambda\sum_{j=1}^{k}\frac{\partial Z_\varkappa}{\partial z_j}z'_j-\lambda z'_\varkappa\sum_{j=1}^{k}\frac{\partial\Theta}{\partial z_j}z'_j+\sum_{j=1}^{k}\left(z'_\varkappa\frac{\partial\Theta}{\partial z_{k+j}}-\frac{\partial Z_\varkappa}{\partial z_{k+j}}\right)$$
$$\times(a_{j1}z_1+\ldots+a_{jk}z_k)\Big]+O(\mu^2) \quad (\varkappa=1, \ldots, k). \tag{2.9}$$

All functions and partial derivatives in (2.9) are evaluated at $\rho=0$, and the components $z_{k+1}, \ldots, z_{2k}$ of the vector $\mathbf{z}$ are replaced by $\lambda z'_1, \ldots, \lambda z'_k$ (primes denote derivatives with respect to ϑ). We also took into account that the derivative of $W(\mathbf{z})$ is zero

$$\sum_{j=1}^{k}\frac{\partial W}{\partial z_j}z'_j-\frac{1}{\lambda}\sum_{j=1}^{k}\frac{\partial W}{\partial z_{k+j}}(a_{j1}z_1+\ldots+a_{jk}z_k)=0,$$

because integral (2.2) implies that $W(\mathbf{z})$ is the integral of the unperturbed (for $\mu=0$) system (2.8).

Note that if $U, V_1, \ldots, V_k$ are independent of $\dot{v}_1, \ldots, \dot{v}_k$ (this is the case in a number of applications), the last sum in brackets in (2.9) is zero.

The case of an initial Lyapunov system of second order, in particular, a mechanical system with one degree of freedom, will not be considered here. The existence of the first integral makes it possible

to reduce the integration of the system to quadratures. The periodic solution for a conservative holonomic system with one degree of freedom depends on two constants of integration (for example, the initial value of the basic coordinate and that of its derivative) and therefore is the general solution. This was discussed extensively in the literature. We only note here that Lyapunov's substitution (1.4) (or (2.3)), series (1.8), and the second equation of (1.5) can yield the most suitable form of the periodic solution. The reader is advised to refer to Malkin's monograph ([111b], Ch. VII, § 4).

§ 2. On the Poincaré Method of Finding Periodic Solutions of Nonautonomous Quasilinear Systems

The method of small parameter is widely used in the theory of nonlinear oscillations. Originating from the theory of small perturbations in celestial mechanics, the method was first described in Poincaré's classical treatise ([188a], vol. 1, Ch. III). Further developments were connected for the most part with Russian schools of research. Foremost among Russian authors are Andronov and Vitt [4, 5], Bulgakov [31], and Malkin [111a, b].

2.1. Differential equations of the generating solution and first corrections. With no pretensions to giving new results, we wish to show that for systems with many degrees of freedom it is very helpful to use the apparatus of matrix theory and some elementary formulations of operator theory. We consider a nonlinear system of k second-order differential equations expressed as a vector equation

$$\mathbf{M}\frac{d^2\mathbf{v}}{dt^2}+\mathbf{Q}_0\frac{d\mathbf{v}}{dt}+\mathbf{P}_0\mathbf{v}=\mathbf{f}(t)+\mu\mathbf{g}(t,\ \mathbf{v},\ \dot{\mathbf{v}},\ \mu), \tag{1.1}$$

where $\mathbf{M}$, $\mathbf{Q}_0$, and $\mathbf{P}_0$ are constant real $k \times k$ matrices ($\mathbf{M}$ and $\mathbf{P}_0$ are symmetric, and $\mathbf{Q}_0$ is skew-symmetric), $\mathbf{f}(t)$ is an integrable piecewise-continuous vector-function of period $T = 2\pi/\omega$, $\mathbf{g}(t, \mathbf{v}, \dot{\mathbf{v}}, \mu)$ is an analytic vector-function of μ for $\mu = 0$, and T is periodic in t and has sufficiently many derivatives with respect to $\mathbf{v}$ and $\dot{\mathbf{v}}$ in the relevant range of variables.

We wish to find T-periodic solutions of equation (1.1) that are analytic in μ for $\mu = 0$

$$\mathbf{v}(t,\ \mu) = \mathbf{v}^{(0)}(t) + \mu\mathbf{v}^{(1)}(t) + \mu^2\mathbf{v}^{(2)}(t) + \dots. \tag{1.2}$$

Substituting this into (1.1), we see that the generating solution $\mathbf{v}^{(0)}(t)$ must satisfy the following system of linear differential equations with constant coefficients

$$\mathbf{M}\frac{d^2\mathbf{v}^{(0)}}{dt^2}+\mathbf{Q}_0\frac{d\mathbf{v}^{(0)}}{dt}+\mathbf{P}_0\mathbf{v}^{(0)}=\mathbf{f}(t), \tag{1.3}$$

and the first and second corrections satisfy the systems

$$\mathbf{M}\frac{d^2\mathbf{v}^{(1)}}{dt^2}+\mathbf{Q}_0\frac{d\mathbf{v}^{(1)}}{dt}+\mathbf{P}_0\mathbf{v}^{(1)}=\mathbf{g}(t,\,\mathbf{v}^{(0)},\,\dot{\mathbf{v}}^{(0)},\,0),$$

$$\mathbf{M}\frac{d^2\mathbf{v}^{(2)}}{dt^2}+\mathbf{Q}_0\frac{d\mathbf{v}^{(2)}}{dt}+\mathbf{P}_0\mathbf{v}^{(2)}=\left(\frac{\partial\mathbf{g}}{\partial\mathbf{v}}\right)_0\mathbf{v}^{(1)}+\left(\frac{\partial\mathbf{g}}{\partial\dot{\mathbf{v}}}\right)_0\dot{\mathbf{v}}^{(1)}+\left(\frac{\partial\mathbf{g}}{\partial\mu}\right)_0, \tag{1.4}$$

where the zero subscript indicates that the partial derivatives in question are to be evaluated for $\mu = 0$, $\mathbf{v} = \mathbf{v}^{(0)}$, $\dot{\mathbf{v}} = \dot{\mathbf{v}}^{(0)}$. The symbols $\partial\mathbf{g}/\partial\mathbf{v}$ and $\partial\mathbf{g}/\partial\dot{\mathbf{v}}$ denote the matrices of the corresponding partial derivatives

$$\frac{\partial\mathbf{g}}{\partial\mathbf{v}}=\left\|\frac{\partial g_i}{\partial v_\varkappa}\right\|_1^k,\quad \frac{\partial\mathbf{g}}{\partial\dot{\mathbf{v}}}=\left\|\frac{\partial g_i}{\partial\dot{v}_\varkappa}\right\|_1^k,\quad \mathbf{v}=\begin{pmatrix}v_1\\ \vdots\\ v_k\end{pmatrix},\quad \mathbf{g}=\begin{pmatrix}g_1\\ \vdots\\ g_k\end{pmatrix}.$$

The homogeneous system corresponding to (1.3) is

$$\mathbf{M}\frac{d^2\mathbf{v}^{(0)}}{dt^2}+\mathbf{Q}_0\frac{d\mathbf{v}^{(0)}}{dt}+\mathbf{P}_0\mathbf{v}^{(0)}=\mathbf{0}. \tag{1.5}$$

We let $\lambda_{\pm 1}, \ldots, \lambda_{\pm k}$ denote the roots of the characteristic equation $(\lambda_{-\nu} = \bar{\lambda}_\nu)$

$$\det(\lambda^2\mathbf{M} + \lambda\mathbf{Q}_0 + \mathbf{P}_0) = 0 \tag{1.6}$$

and $\mathbf{a}_\nu$ the corresponding eigenvectors

$$(\lambda_\nu^2\mathbf{M} + \lambda_\nu\mathbf{Q}_0 + \mathbf{P}_0)\,\mathbf{a}_\nu = \mathbf{0} \quad (\nu = \pm 1, \ldots, \pm k).$$

We shall consider two cases.

2.2. Nonresonant case. System (1.5) has no T-periodic solutions. A necessary and sufficient condition for this is that

$$\lambda_\nu \neq ip\omega \quad (\nu = \pm 1, \ldots, \pm k; \quad p = 0, \pm 1, \pm 2, \ldots). \tag{2.1}$$

In this so-called nonresonant case system (1.3) (and similarly (1.4) and the following) has a unique T-periodic solution (see below). According to the Poincaré theorem ([188a], vol. 1, Ch. III, see also [111a]), series (1.2) converges for $\mu = 0$, that is, the initial nonlinear system of equations (1.1) has a unique T-periodic solution that is analytic in μ for $\mu = 0$.

We now derive an expression for a T-periodic solution of system (1.3). We assume for the sake of simplicity that $\mathbf{Q}_0 = \mathbf{0}$ and, without losing generality, let $\mathbf{M}_0 = \mathbf{I}_k$. This yields the equation

$$\frac{d^2\mathbf{v}^{(0)}}{dt^2}+\mathbf{P}_0\mathbf{v}^{(0)}=\mathbf{f}(t) \quad (\mathbf{P}_0^\tau=\mathbf{P}_0) \tag{2.2}$$

in the form of a first-order vector equation

$$\frac{d\mathbf{x}}{dt}=\mathbf{A}\mathbf{x}+\mathbf{h}(t),\quad \mathbf{x}=\begin{pmatrix}\mathbf{v}^{(0)}\\ \dot{\mathbf{v}}^{(0)}\end{pmatrix},\quad \mathbf{A}=\begin{pmatrix}0 & \mathbf{I}_k\\ -\mathbf{P}_0 & 0\end{pmatrix},\quad \mathbf{h}=\begin{pmatrix}0\\ \mathbf{f}\end{pmatrix}. \tag{2.3}$$

A solution of system (2.3) with initial conditions is given by the formula (see, for example [80], II, 1.4)

$$\mathbf{x}(t) = e^{t\mathbf{A}}\left[\mathbf{x}(0) + \int_0^t e^{-s\mathbf{A}}\mathbf{h}(s)\,ds\right],$$

which yields

$$\mathbf{x}(t+T) = e^{T\mathbf{A}}e^{t\mathbf{A}}\left[\mathbf{x}(0) + \int_0^t e^{-s\mathbf{A}}\mathbf{h}(s)\,ds + \int_t^{t+T} e^{-s\mathbf{A}}\mathbf{h}(s)\,ds\right]$$

$$= e^{T\mathbf{A}}\left[\mathbf{x}(t) + \int_t^{t+T} e^{(t-s)\mathbf{A}}\mathbf{h}(s)\,ds\right].$$

Replacing s in the last integral by $\tau = T + t - s$, we write the necessary and sufficient condition of periodicity, $\mathbf{x}(t+T) = \mathbf{x}(t)$,

$$e^{T\mathbf{A}}\left[\mathbf{x}(t) + \int_0^T e^{(\tau-T)\mathbf{A}}\mathbf{h}(t-\tau)\,d\tau\right] = \mathbf{x}(t).$$

The matrix $\mathbf{I}_{2k} - e^{T\mathbf{A}}$ is nonsingular by virtue of (2.1), and hence the unique T-periodic solution of system (2.3) is given by

$$\mathbf{x}(t) = [\mathbf{I}_{2k} - e^{T\mathbf{A}}]^{-1}\int_0^T e^{\tau\mathbf{A}}\mathbf{h}(t-\tau)\,d\tau. \qquad (2.4)$$

From (2.3) we obtain for the matrizant $e^{t\mathbf{A}}$

$$e^{t\mathbf{A}} = \begin{pmatrix} \cos\left(t\sqrt{\mathbf{P}_0}\right) & \left(\sqrt{\mathbf{P}_0}\right)^{-1}\sin\left(t\sqrt{\mathbf{P}_0}\right) \\ -\sqrt{\mathbf{P}_0}\sin\left(t\sqrt{\mathbf{P}_0}\right) & \cos\left(t\sqrt{\mathbf{P}_0}\right) \end{pmatrix},$$

where the above matrix-functions are given by expressions derived in [80], (2.3), and Chapter I, Subsection 2.1. For the first multiplier of formula (2.4) we find

$$[\mathbf{I}_{2k} - e^{T\mathbf{A}}]^{-1} = \frac{1}{2}\begin{pmatrix} \mathbf{I}_k & \left(\sqrt{\mathbf{P}_0}\right)^{-1}\cot\left(\frac{1}{2}T\sqrt{\mathbf{P}_0}\right) \\ -\sqrt{\mathbf{P}_0}\cot\left(\frac{1}{2}T\sqrt{\mathbf{P}_0}\right) & \mathbf{I}_k \end{pmatrix},$$

which can be verified by multiplying by $\mathbf{I}_{2k} - \exp(T\mathbf{A})$. If a matrix $\sqrt{\mathbf{P}_0}$ has a sufficiently small norm, we obtain

$$\cot\left(\frac{1}{2}T\sqrt{\mathbf{P}_0}\right) = \frac{2}{T}\left(\sqrt{\mathbf{P}_0}\right)^{-1} - \frac{1}{6}T\sqrt{\mathbf{P}_0} - \frac{1}{3^2\cdot 5}\left(\frac{1}{2}T\sqrt{\mathbf{P}_0}\right)^3 - \dots.$$

General trigonometric formulas hold true for the trigonometric matrix-functions just introduced (see [80], § 1, 2). Therefore, provided (2.1) is satisfied, (2.4) yields the following formulas for the unique T-periodic solution of system (2.2) and its derivative

$$\mathbf{v}^{(0)}(t)=\frac{1}{2}\int_0^T (\sqrt{\mathbf{P}_0})^{-1}\sin(\tau\sqrt{\mathbf{P}_0})\,\mathbf{f}(t-\tau)\,d\tau$$

$$+\frac{1}{2}(\sqrt{\mathbf{P}_0})^{-1}\cot\left(\frac{1}{2}T\sqrt{\mathbf{P}_0}\right)\int_0^T \cos(\tau\sqrt{\mathbf{P}_0})\,\mathbf{f}(t-\tau)\,d\tau,$$

$$\dot{\mathbf{v}}^{(0)}(t)=-\frac{1}{2}\cot\left(\frac{1}{2}T\sqrt{\mathbf{P}_0}\right)\int_0^T \sin(\tau\sqrt{\mathbf{P}_0})\,\mathbf{f}(t-\tau)\,d\tau$$

$$+\frac{1}{2}\int_0^T \cos(\tau\sqrt{\mathbf{P}_0})\,\mathbf{f}(t-\tau)\,d\tau. \quad (2.5)$$

2.3. Resonant case. We suppose that the homogeneous system (1.5) has $2d$ ($1\leqslant d\leqslant k$) linearly independent T-periodic solutions. Then we have $2d$ equalities

$$\lambda_j=ip_j\omega \quad \left(j=\pm 1,\ \ldots,\ \pm d;\quad p_{-j}=-p_j,\ \omega=\frac{2\pi}{T}\right),$$

and for the other roots of the characteristic equation (1.6) the inequalities

$$\lambda_h\neq ip\omega \quad (h=\pm(d+1),\ \ldots,\ \pm k),$$

where p_j and p are integers. We write the $2d$ T-periodic vector-solutions of system (1.5) in complex form

$$\mathbf{v}_{j0}^{(\pm)}=e^{\pm ip_j\omega t}\mathbf{a}_j \quad (j=1,\ \ldots,d). \quad (3.1)$$

Let us establish the necessary and sufficient conditions for the existence of T-periodic solutions of system (1.3) in the resonant case. Let $\mathbf{u}(t)$ and $\mathbf{w}(t)$ be vector-functions with integrable piecewise-continuous second derivatives on $[0, T]$, the end conditions of the vector-functions and their first derivatives being

$$\mathbf{u}(T)=\mathbf{u}(0),\qquad \dot{\mathbf{u}}(T)=\dot{\mathbf{u}}(0),$$

$$\mathbf{w}(T)=\mathbf{w}(0),\qquad \dot{\mathbf{w}}(T)=\dot{\mathbf{w}}(0).$$

Then*

$$\int_0^T (\mathbf{M}\ddot{\mathbf{u}}+\mathbf{Q}_0\dot{\mathbf{u}}+\mathbf{P}_0\mathbf{u},\ \mathbf{w})\,dt=\int_0^T (\mathbf{u},\ \mathbf{M}\ddot{\mathbf{w}}+\mathbf{Q}_0\dot{\mathbf{w}}+\mathbf{P}_0\mathbf{w})\,dt. \tag{3.2}$$

Indeed, from $(\mathbf{A}\mathbf{x},\ \mathbf{y}) = (\mathbf{x},\ \mathbf{A}^*\mathbf{y})$ we obtain

$$\int_0^T (\mathbf{M}\ddot{\mathbf{u}}+\mathbf{Q}_0\dot{\mathbf{u}}+\mathbf{P}_0\mathbf{u},\ \mathbf{w})\,dt$$

$$=\int_0^T (\ddot{\mathbf{u}},\ \mathbf{M}\mathbf{w})\,dt-\int_0^T (\dot{\mathbf{u}},\ \mathbf{Q}_0\mathbf{w})\,dt+\int_0^T (\mathbf{u},\mathbf{P}_0\mathbf{w})\,dt.$$

Integrating by parts, once in the second integral and twice in the first, we obtain formula (3.2). Suppose that system (1.1) has a periodic solution $\mathbf{v}\,(t,\ \mu)$. Substituting this solution and any one of the $2d$ functions of (3.1) into (3.2), we obtain the following equality which holds identically in μ

$$\int_0^T (\mathbf{f}\,(t)+\mu\mathbf{g}\,(t,\ \mathbf{v},\ \dot{\mathbf{v}},\ \mu),\ \mathbf{v}_{j0}^{(\pm)})\,dt=0 \quad (j=1,\ \ldots,d), \tag{3.3}$$

since

$$\mathbf{M}\ddot{\mathbf{v}}_{j0}^{(\pm)}+\mathbf{Q}_0\dot{\mathbf{v}}_{j0}^{(\pm)}+\mathbf{P}_0\mathbf{v}_{j0}^{(\pm)}=\mathbf{0}, \quad (j=1,\ \ldots,d).$$

Setting $\mu = 0$ in (3.3), we obtain

$$\int_0^T (\mathbf{f}\,(t),\ \mathbf{v}_{j0}^{(\pm)})\,dt=0 \quad (j=1,\ \ldots,d).$$

Assuming that these equalities hold, let us subtract them from identities (3.3). Dividing by μ, we obtain the identities

$$\int_0^T (\mathbf{g}\,(t,\ \mathbf{v},\ \dot{\mathbf{v}},\ \mu),\ \mathbf{v}_{j0}^{(\pm)})\,dt=0 \quad (j=1,\ \ldots,d). \tag{3.4}$$

---

* If

$$\mathbf{x}=\begin{pmatrix}\xi_1\\ \cdot\\ \cdot\\ \cdot\\ \xi_k\end{pmatrix},\qquad \mathbf{y}=\begin{pmatrix}\eta_1\\ \cdot\\ \cdot\\ \cdot\\ \eta_k\end{pmatrix},$$

then $(\mathbf{x},\ \mathbf{y}) = \xi_1\bar{\eta}_1 + \ldots + \xi_k\bar{\eta}_k$ (bars over scalars indicate conjugate complex quantities).

Setting $\mu = 0$ and substituting $\mathbf{v}_{j0}^{(\pm)}$ from (3.1), we see that in the resonant case the generating solution $\mathbf{v}^{(0)}(t)$ must satisfy the following $2d$ conditions

$$\psi_j \equiv \int_0^T e^{-ip_j\omega t}(\mathbf{g}(t, \mathbf{v}^{(0)}, \dot{\mathbf{v}}^{(0)}, 0), \mathbf{a}_j)\,dt = 0 \tag{3.5}$$

$$(j = \pm 1, \ldots, \pm d).$$

We shall look for a generating solution $\mathbf{v}^{(0)}(t)$ in the form

$$\mathbf{v}^{(0)} = \mathbf{w}^{(0)}(t) + \sum_{j=-d}^{d} \zeta_j e^{ip_j\omega t}\mathbf{a}_j$$

$$(\zeta_{-j} = \bar{\zeta}_j,\ p_{-j} = -p_j),$$

where $\mathbf{w}^{(0)}(t)$ is a partial periodic solution of system (1.3). Conditions (3.5) now constitute a system of $2d$ equations* for the complex numbers $\zeta_{-d}, \ldots, \zeta_d$; let us call them equations for "generating amplitudes". Let $\zeta_{-d}^{(0)}, \ldots, \zeta_d^{(0)}$ denote a solution of system (3.5). It can be shown that if the Jacobian of system (3.5) does not vanish, i.e.

$$\frac{D(\psi_{-d}, \ldots, \psi_d)}{D(\zeta_{-d}^{(0)}, \ldots, \zeta_d^{(0)})} \neq 0,$$

then solution (1.2) exists and is analytic in μ for $\mu = 0$.

Equating terms in $\mu, \mu^2, \ldots$ in identities (3.4), we get

$$\chi_j \equiv \int_0^T e^{-ip_j\omega t}\left(\left(\frac{\partial \mathbf{g}}{\partial \mathbf{v}}\right)_0 \mathbf{v}^{(1)} + \left(\frac{\partial \mathbf{g}}{\partial \dot{\mathbf{v}}}\right)_0 \dot{\mathbf{v}}^{(1)} + \left(\frac{\partial \mathbf{g}}{\partial \mu}\right)_0, \mathbf{a}_j\right) dt = 0 \tag{3.6}$$

$$(j = \pm 1, \ldots, \pm d),$$

where the zero subscript means that the partial derivatives are evaluated at $\mu = 0$, $\mathbf{v} = \mathbf{v}^{(0)}(t)$, $\dot{\mathbf{v}} = \dot{\mathbf{v}}^{(0)}(t)$.

2.4. Variational equations for periodic unperturbed motion. Letting (1.2) denote unperturbed motion, we will write differential equations of perturbed motion in the form

$$\mathbf{v} = \mathbf{v}(t, \mu) + \mathbf{y}.$$

Dropping second-order terms in $\mathbf{y}$, we obtain the variational equation

$$\mathbf{M}\frac{d^2\mathbf{y}}{dt^2} + \mathbf{Q}_0\frac{d\mathbf{y}}{dt} + \mathbf{P}_0\mathbf{y} = \mu\left(\frac{\partial \mathbf{g}}{\partial \mathbf{v}}\right)\mathbf{y} + \mu\left(\frac{\partial \mathbf{g}}{\partial \dot{\mathbf{v}}}\right)\frac{d\mathbf{y}}{dt},$$

* These equations are derived from the general Fredholm theorem in operator theory.

where unperturbed motion (1.2), $\mathbf{v} = \mathbf{v}(t, \mu)$, $\dot{\mathbf{v}} = \dot{\mathbf{v}}(t, \mu)$, is substituted into the partial derivatives after differentiation. We denote matrix-functions by

$$\mu\left(\frac{\partial g}{\partial \mathbf{v}}\right) = -\mu \mathbf{P}(t, \mu), \quad \mu\left(\frac{\partial g}{\partial \dot{\mathbf{v}}}\right) = -\mu \mathbf{Q}(t, \mu).$$

We expand these as power series in μ

$$\mu \mathbf{P}(t, \mu) = \mu \mathbf{P}_1(t) + \mu^2 \mathbf{P}_2(t) + \ldots,$$
$$\mu \mathbf{Q}(t, \mu) = \mu \mathbf{Q}_1(t) + \mu^2 \mathbf{Q}_2(t) + \ldots,$$

where

$$\mathbf{P}_1(t) = -\left(\frac{\partial g}{\partial \mathbf{v}}\right)_0, \quad \mathbf{Q}_1(t) = -\left(\frac{\partial g}{\partial \dot{\mathbf{v}}}\right)_0, \tag{4.1}$$

and the zero subscript means that the derivatives are evaluated at $\mu = 0$, $\mathbf{v} = \mathbf{v}^{(0)}(t)$, $\dot{\mathbf{v}} = \dot{\mathbf{v}}^{(0)}(t)$. We write the vector equation corresponding to the variational equation as

$$\mathbf{M}\frac{d^2\mathbf{y}}{dt^2} + [\mathbf{Q}_0 + \mu \mathbf{Q}_1(t) + \mu^2 \mathbf{Q}_2(t) + \ldots]\frac{d\mathbf{y}}{dt}$$
$$+ [\mathbf{P}_0 + \mu \mathbf{P}_1(t) + \mu^2 \mathbf{P}_2(t) + \ldots]\mathbf{y} = 0. \tag{4.2}$$

We thus have a system of linear differential equations with T-periodic coefficients.

2.5. Case of distinct multipliers of unperturbed system of variational equations. We consider the variational equation (4.2) for $\mu = 0$

$$\mathbf{M}\frac{d^2\mathbf{y}}{dt^2} + \mathbf{Q}_0\frac{d\mathbf{y}}{dt} + \mathbf{P}_0\mathbf{y} = 0. \tag{5.1}$$

Under the conditions $\mathbf{M}^\tau = \mathbf{M} > 0$, $\mathbf{P}_0^\tau = \mathbf{P}_0 > 0$, $\mathbf{Q}_0^\tau = -\mathbf{Q}_0$, all roots of its characteristic equation (1.6) are pure imaginary

$$\lambda_\nu = i\omega_\nu \quad (\nu = \pm 1, \ldots, \pm k; \quad \omega_{-\nu} = -\omega_\nu). \tag{5.2}$$

Let us assume that all the multipliers of system (5.1) corresponding to period T

$$\rho_\nu = e^{i\omega_\nu T} \quad (\nu = \pm 1, \ldots, \pm k)$$

are distinct, $\rho_\chi \neq \rho_\nu$, that is,

$$\omega_\chi \not\equiv \omega_\nu \quad \left(\mathrm{mod}\ \omega \equiv \frac{2\pi}{T}\right) \quad (\chi, \nu = \pm 1, \ldots, \pm k).$$

In other words, each congruence class of the numbers ω_ν modulo $2\pi/T$ consists of a single number. The characteristic exponents of the system of variational equations are ([80], IV, 3.4)

$$\alpha_j(\mu) = i\omega_j - i\sigma_{jj}\,\mathrm{sign}\, j \cdot \mu + O(\mu^2) \quad (j = \pm 1, \ldots, \pm k),$$

where*

$$\det[-\omega_j^2\mathbf{M}+i\omega_j\mathbf{Q}_0+\mathbf{P}_0]=0,$$

$$\sigma_{jj}=-([\mathbf{P}_1^{(0)}+i\omega_j\mathbf{Q}_1^{(0)}]\,\mathbf{a}_j,\ \mathbf{a}_j),$$

$$\mathbf{P}_1^{(0)}=\frac{1}{T}\int_0^T \mathbf{P}_1(t)\,dt,\quad \mathbf{Q}_1^{(0)}=\frac{1}{T}\int_0^T \mathbf{Q}_1(t)\,dt \tag{5.3}$$

and $\mathbf{a}_j$ are normalized eigenvectors

$$(-\omega_j^2\mathbf{M}+i\omega_j\mathbf{Q}_0+\mathbf{P}_0)\,\mathbf{a}_j=0,$$

$$([2\omega_j\mathbf{M}-i\mathbf{Q}_0]\,\mathbf{a}_j,\ \mathbf{a}_j)=\operatorname{sign} j\quad (j=\pm 1,\ \ldots,\ \pm k). \tag{5.4}$$

In the case under consideration, for the real parts of the characteristic exponents of the system of variational equations (4.1) we obtain

$$\operatorname{Re}\alpha_j(\mu)=-\operatorname{sign} j\cdot\operatorname{Im}([\mathbf{P}_1^{(0)}+i\omega_j\mathbf{Q}_1^{(0)}]\,\mathbf{a}_j,\ \mathbf{a}_j)\,\mu+O(\mu^2)$$

$$(j=\pm 1,\ \ldots,\ \pm k).$$

Consequently, *if*

$$\operatorname{sign} j\cdot\operatorname{Im}([\mathbf{P}_1^{(0)}+i\omega_j\mathbf{Q}_1^{(0)}]\,\mathbf{a}_j,\ \mathbf{a}_j)>0\quad (j=\pm 1,\ \ldots,\ \pm k), \tag{5.5}$$

that is, for all j, then unperturbed motion (1.2) *is asymptotically stable* [108a], [40a] *for sufficiently small positive* μ; *if, however,*

$$\operatorname{sign} j'\cdot\operatorname{Im}([\mathbf{P}_1^{(0)}+i\omega_{j'}\mathbf{Q}_1^{(0)}]\,\mathbf{a}_{j'},\ \mathbf{a}_{j'})<0 \tag{5.6}$$

for at least one scalar product (for some j' in the sequence $\pm 1, \ldots$ $\ldots, \pm k$*), then unperturbed motion* (1.2) *is unstable* [108a], [40a] *for sufficiently small positive* μ.

Remark. We assume that in the case of distinct multipliers under consideration

$$(\mathbf{P}_1^{(0)}\mathbf{a}_j,\ \mathbf{a}_j)\quad (j=\pm 1,\ \ldots,\ \pm k)$$

are *real.* This is certainly true for the trivial condition $\mathbf{P}_1^{(0)}=\mathbf{0}$ as well as for $\mathbf{Q}_0=\mathbf{0}$. In the latter case $\mathbf{a}_1, \ldots, \mathbf{a}_k$ are real as eigenvectors of the symmetric matrix $\mathbf{M}^{-1}\mathbf{P}_0$, and $\mathbf{a}_{-j}=\mathbf{a}_j$ ($j=1, \ldots, k$). Since $\omega_j \operatorname{sign} j>0$ (see (5.2)),

$$\operatorname{Re}\alpha_j(\mu)=-\operatorname{Re}(\mathbf{Q}_1^{(0)}\mathbf{a}_j,\ \mathbf{a}_j)\,\mu+O(\mu^2)\quad (j=\pm 1,\ \ldots,\ \pm k).$$

---

* We remind the reader that $(\mathbf{x},\mathbf{y})=\sum \xi_\varkappa\bar{\eta}_\varkappa$ (see the footnote to Subsection 2.3).

Hence, *if for all j*

$$\vartheta_j \equiv -\operatorname{Re}(\mathbf{Q}_1^{(0)}\mathbf{a}_j,\ \mathbf{a}_j)$$
$$= \frac{1}{T}\operatorname{Re}\left(\int_0^T \left(\frac{\partial \mathbf{g}}{\partial \dot{\mathbf{v}}}\right)_0 dt \cdot \mathbf{a}_j,\ \mathbf{a}_j\right) < 0 \quad (j = \pm 1,\ \ldots,\ \pm k), \tag{5.5'}$$

then unperturbed motion (1.2) *is asymptotically stable for sufficiently small positive* μ, *and if for at least one of the scalar products, then it is unstable for sufficiently small positive* μ

$$\vartheta_{j'} \equiv \frac{1}{T}\operatorname{Re}\left(\int_0^T \left(\frac{\partial \mathbf{g}}{\partial \dot{\mathbf{v}}}\right)_0 dt \cdot \mathbf{a}_{j'},\ \mathbf{a}_{j'}\right) > 0. \tag{5.6'}$$

If $\mathbf{Q}_0 = 0$, *then it is sufficient to test conditions* (5.5′) *and* (5.6′) *only for positive subscripts*; *the sign* Re *is automatically dropped because* $(\mathbf{Q}_1^{(0)}\mathbf{a}_j,\ \mathbf{a}_j)$ *are real* $(j = 1,\ \ldots,\ k)$.

2.6. Case of multiple multipliers ([80], V, 3.4). We assume that system (5.1) has an r-fold multiplier ρ_0 corresponding to period T

$$\rho_0 = e^{i\omega^{(0)}T}.$$

Let $\omega_{j_1},\ \ldots,\ \omega_{j_r}$ be the class of roots of equation (5.3) corresponding to this multiplier, i.e.

$$\omega_j \equiv \omega^{(0)} \quad \left(\operatorname{mod} \omega \equiv \frac{2\pi}{T}\right) \quad (j = j_1,\ \ldots,\ j_r).$$

System (4.2) thus has for $\mu = 0$ a multiple characteristic exponent $i\omega^{(0)}$. We define integers m_j by

$$\omega_j = \omega^{(0)} + m_j \frac{2\pi}{T} \quad (j = j_1,\ \ldots,\ j_r).$$

We expand the matrix-functions (see (4.1))

$$\mathbf{P}_1(t) = -\left(\frac{\partial \mathbf{g}}{\partial \mathbf{v}}\right)_0, \quad \mathbf{Q}_1(t) = -\left(\frac{\partial \mathbf{g}}{\partial \dot{\mathbf{v}}}\right)_0$$

in complex Fourier series

$$\mathbf{P}_1(t) \sim \sum_{m=-\infty}^{\infty} \exp\left(im\frac{2\pi}{T}\right)\mathbf{P}_1^{(m)}, \quad \mathbf{Q}_1(t) \sim \sum_{m=-\infty}^{\infty} \exp\left(im\frac{2\pi}{T}\right)\mathbf{Q}_1^{(m)}.$$

We define σ_{jh} by

$$\sigma_{jh} = -([\mathbf{P}_1^{(m_h - m_j)} + i\omega_j\mathbf{Q}_1^{(m_h - m_j)}]\,\mathbf{a}_j,\ \mathbf{a}_h) \tag{6.1}$$
$$(j,\ h = j_1,\ \ldots,\ j_r),$$

where $\mathbf{a}_j$ are normalized by conditions (5.4), and

$$\mathbf{P}_1^{(m_h-m_j)}=\frac{1}{T}\int\limits_0^T e^{-i(\omega_h-\omega_j)t}\mathbf{P}_1(t)\,dt,$$

$$\mathbf{Q}_1^{(m_h-m_j)}=\frac{1}{T}\int\limits_0^T e^{-i(\omega_h-\omega_j)t}\mathbf{Q}_1(t)\,dt$$

We introduce the numbers

$$\gamma_{jh}=\begin{cases} 0 & j\neq h,\\ 1 & j=h>0,\\ -1 & j=h<0\end{cases}\qquad (j,\ h=j_1,\ \ldots,\ j_r)$$

and build the equation

$$\det\|\sigma_{jh}-i\chi\gamma_{jh}\|_{j,\ h=j_1,\ \ldots,\ j_r}=0. \tag{6.2}$$

If the real parts of the roots of this equation, χ_{j_1}, . . ., χ_{j_r} *are negative and this is true for all the multipliers of system* (5.1), *then unperturbed motion* (1.2) *is asymptotically stable* [108a], [40a] *for sufficiently small positive* μ. *If the real part of at least one of the roots of equation* (6.2) *is positive in one of the classes, then unperturbed motion* (1.2) *is unstable* [108a], [40a] *for sufficiently small positive* μ.

The following formula holds for r characteristic exponents of the system of variational equations (4.2) transformed to $i\omega^{(0)}$ for $\mu=0$

$$\alpha_j(\mu)=i\omega^{(0)}+\chi_j\mu+O(\mu^{1+\delta})\quad (\delta>0;\ j=j_1,\ \ldots,\ j_r),$$

where χ_j are roots of equation (6.2).

2.7. Examples.

Example 1. We consider the scalar Van der Pol equation

$$\ddot{\xi}+\sigma^2\xi=l\sin t+\mu(1-\xi^2)\dot{\xi}\quad (\sigma>0,\ l>0),$$

which was investigated in detail by Andronov and Vitt ([4], pp. 70-84). The roots of the characteristic equation (1.6) are $\lambda=\pm i\sigma$. We shall study the two possible cases.

(a) *Nonresonant case* (σ *is not a natural number*). Equation (1.3) is written in the form

$$\ddot{\xi}^{(0)}+\sigma^2\xi^{(0)}=l\sin t$$

and it has a unique 2π-periodic ($T=2\pi$) generating solution

$$\xi^{(0)}=\frac{l}{\sigma^2-1}\sin t.$$

Using the first equation of (1.4), which in this case is

$$\ddot{\xi}^{(1)}+\sigma^2\xi^{(1)}=\left[1-\frac{l^2\sin^2 t}{(\sigma^2-1)^2}\right]\frac{l}{\sigma^2-1}\cos t,$$

we determine the first correction

$$\xi^{(1)}=\frac{l}{(\sigma^2-1)^2}\left[1-\frac{l^2}{4(\sigma^2-1)^2}\right]\cos t+\frac{l^3}{4(\sigma^2-1)^3(\sigma^2-9)}\cos 3t.$$

All subsequent terms in series (1.2) are also uniquely determined from the second equation of (1.4) and from subsequent equations. The required solution of (1.2) is

$$\xi(t,\ \mu)=\frac{l\sin t}{\sigma^2-1}+\frac{\mu l}{(\sigma^2-1)^2}\left\{\left[1-\frac{l^2}{4(\sigma^2-1)^2}\right]\cos t\right.$$
$$\left.+\frac{l^2\cos 3t}{4(\sigma^2-1)(\sigma^2-9)}\right\}+O(\mu^2).$$

To check for stability, we calculate ϑ using formula (5.5′)

$$\vartheta=\frac{1}{2\pi}\int\limits_0^{2\pi}\left\{\frac{\partial}{\partial\dot{\xi}}\left[(1-\xi^2)\,\dot{\xi}\right]\right\}_0 dt$$

$$=\frac{1}{2\pi}\int\limits_0^{2\pi}\left[1-\frac{l^2}{(\sigma^2-1)^2}\sin^2 t\right]dt=1-\frac{l^2}{2(\sigma^2-1)^2},$$

and, according to formulas (5.5′) and (5.6′), we conclude that for sufficiently small μ and

$$l^2>2(\sigma^2-1)^2 \tag{7.1}$$

the above 2π-periodic solution $\xi(t,\mu)$ is asymptotically stable; it is unstable if the sign of the inequality is reversed.

(b) *Resonant case* ($\sigma=p$, *where p is a natural number*). Equation (1.3) takes the form

$$\ddot{\xi}^{(0)}+p^2\xi^{(0)}=l\sin t$$

and it has a 2π-periodic solution for any $p\geqslant 2$; we shall assume henceforth that $p\geqslant 2$. From Subsection 2.3,

$$\xi^{(0)}=\frac{l}{p^2-1}\sin t+\zeta_1\cos pt+\zeta_2\sin pt.$$

After obvious operations, the left-hand sides of equations (3.5) are determined and the equations themselves are written as

$$\psi_1=-\frac{\pi}{4}p\zeta_2\left[\zeta_1^2+\zeta_2^2+\frac{2l^2}{(p^2-1)^2}-4\right]=0,$$

$$\psi_2=\frac{\pi}{4}p\zeta_1\left[\zeta_1^2+\zeta_2^2+\frac{2l^2}{(p^2-1)^2}-4\right]=0. \tag{7.2}$$

For any values of the parameters p and l, the equations have the trivial solution

$$\zeta_1=\zeta_2=0,$$

and if $l < \sqrt{2}\,(p^2 - 1)$ they also have the solution

$$\zeta_1 = \zeta_1^{(0)}, \quad \zeta_2 = \zeta_2^{(0)} = \mp \sqrt{4 - \frac{2l^2}{(p^2-1)^2} - \zeta_1^{(0)2}},$$

where $\zeta_1^{(0)}$ is any real number whose absolute value is not greater than $[4 - 2l^2/(p^2 - 1)^2]^{1/2}$. For the nontrivial solution, the Jacobian of Subsection 2.3 vanishes

$$\left.\frac{D(\psi_1, \psi_2)}{D(\zeta_1, \zeta_2)}\right|_{\zeta_1=\zeta_1^{(0)},\ \zeta_2=\zeta_2^{(0)}} = 0$$

and so Subsection 2.3 does not tell us whether there exists a 2π-periodic solution analytic in μ for $\mu = 0$ whose generating solution is

$$\xi^{(0)} = \frac{l}{p^2-1} \sin t + \zeta_1^{(0)} \cos pt + \zeta_2^{(0)} \sin pt.$$

For further details, we refer the reader to [4].

We consider the first solution of equations (7.2), that is, $\zeta_1 = \zeta_2 = 0$. The Jacobian does not vanish for $l \neq \sqrt{2}\,(p^2 - 1)$

$$\left.\frac{D(\psi_1, \psi_2)}{D(\zeta_1, \zeta_2)}\right|_{\zeta_1=\zeta_2=0} = \frac{1}{4}\pi^2 p^2 \left[\frac{l^2}{(p^2-1)^2} - 2\right]^2 \neq 0$$

and, according to Subsection 2.3, there exists a solution analytic in μ for $\mu = 0$. The first equation of (1.4) yields for the first correction and even p

$$\xi^{(1)} = \frac{l}{(p^2-1)^2}\left[1 - \frac{l^2}{4(p^2-1)^2}\right]\cos t + \frac{l^3}{4(p^2-1)^3(p^2-9)} \cos 3t + \eta_1 \cos pt + \eta_2 \sin pt,$$

where η_1 and η_2 are defined by equation (3.6). The stability condition is again (7.1).

Example 2. We consider the system of two differential equations ([71], p. 117)

$$\begin{aligned} \ddot{\eta}_1 + \eta_1 &= \mu\,(1 - \eta_1^2 - \eta_2^2)\,\dot{\eta}_1 + \mu q \cos t, \\ \ddot{\eta}_2 + \omega_2^2 \eta_2 &= \mu\,(1 - \eta_1^2 - \eta_2^2)\,\dot{\eta}_2, \end{aligned} \tag{7.3}$$

where $\mu > 0$, $q < 0$, $\omega_2 > 0$ and $\omega_2 \neq \frac{1}{2} m$ (m is a natural number). The roots of the characteristic equation (1.6) are

$$\lambda_{\mp 1} = \mp i, \quad \lambda_{\mp 2} = \mp \omega_2 i,$$

and since $\omega = 2\pi/T = 1$, we have the resonant case: $p_{-1} = -1$, $p_1 = 1$, and $\lambda_{\mp 2} \neq pi$ (see the beginning of Subsection 2.3). The

generating 2π-periodic solution of system (7.3) will be sought in the form $\eta_2^{(0)} \equiv 0$

$$\eta_1^{(0)} = \zeta_1 e^{it} + \bar{\zeta}_1 e^{-it} = 2\,\mathrm{Re}\,(\zeta_1 e^{it}) = 2\alpha_1 \cos t - 2\beta_1 \sin t,$$

where $\zeta_1 = \alpha_1 + i\beta_1$. The left-hand sides of system (3.5) are

$$\psi_{-1} = 2\pi \left[\frac{1}{2} q - i\bar{\zeta}_1 (1 - \zeta_1 \bar{\zeta}_1) \right],$$

$$\psi_1 = 2\pi \left[\frac{1}{2} q + i\zeta_1 (1 - \zeta_1 \bar{\zeta}_1) \right].$$

Adding and subtracting equations (3.5), we obtain

$$q_1 - 2\beta_1 (1 - \alpha_1^2 - \beta_1^2) = 0, \quad 2i\alpha_1 (1 - \alpha_1^2 - \beta_1^2) = 0.$$

Hence we see that $\alpha_1 = 0$, while β_1 must be a root of the cubic equation

$$f(\beta_1) = \beta_1^3 - \beta_1 + \frac{1}{2} q = 0. \tag{7.4}$$

If $q < -4\sqrt{3}/9$, this equation has only one real root $\beta_1^{(0)} > 2\sqrt{3}/3$; if $-4\sqrt{3}/9 < q < 0$, it has one positive root $\beta_1^{(0)}$ $(1 < \beta_1^{(0)} < 2\sqrt{3}/3)$ and two negative roots β_1^* and β_1^{**} $(\beta_1^* < \beta_1^{**})$. Thus the generating solution of system (7.3) is $\eta_2^{(0)} \equiv 0$,

$$\eta_1^{(0)} = \zeta_1^{(0)} e^{it} + \bar{\zeta}_1^{(0)} e^{-it} = -2\beta_1 \sin t \quad (\zeta_1^{(0)} = i\beta_1), \tag{7.5}$$

where β_1 satisfies equation (7.4). We compute the Jacobian of system (3.5)

$$D = \frac{D(\psi_{-1}, \psi_1)}{D(\bar{\zeta}_1^{(0)}, \zeta_1^{(0)})} = 4\pi^2 \left[(1 - 2\zeta_1^{(0)} \bar{\zeta}_1^{(0)})^2 - \zeta_1^{(0)2} \bar{\zeta}_1^{(0)2} \right] = 4\pi^2 (3\beta_1^4 - 4\beta_1^2 + 1).$$

The Jacobian D vanishes only for $\beta_1 = \mp\sqrt{3}/3$ and $\beta_1 = \mp 1$. Since $\beta_1^{(0)} > 1$, it follows that the Jacobian does not vanish at $\beta = \beta_1^{(0)}$. Further, the case $\beta_1 = \mp 1$ is impossible, since $f(\mp 1) < 0$. Thus D vanishes if and only if $\beta_1 = -\sqrt{3}/3$, i.e. if $q = -4\sqrt{3}/9$. This case requires further investigation and will not be considered here. In all other cases solution (1.3) exists and is analytic in μ for $\mu = 0$. The system of equations (1.4) for the first correction takes the form

$$\ddot{\eta}_1^{(1)} + \eta_1^{(1)} = -2\beta_1^3 \cos 3t, \quad \ddot{\eta}_2^{(1)} + \omega_2^2 \eta_2^{(1)} = 0,$$

whence

$$\eta_1^{(1)} = \gamma_1 e^{it} + \bar{\gamma}_1 e^{-it} + \frac{1}{4} \beta_1^3 \cos 3t, \quad \eta_2^{(1)} \equiv 0.$$

We can now determine the generating amplitude γ_1 of the first correction, which we shall not do here, but go on to consider the stability of solution (1.2).

The variational equations (4.2) for system (7.3) and the generating solution (7.5) are

$$\frac{d^2\mathbf{y}}{dt^2}+[\mu\mathbf{Q}_1(t)+\ldots]\frac{d\mathbf{y}}{dt}+[\mathbf{P}_0+\mu\mathbf{P}_1(t)+\ldots]\mathbf{y}=\mathbf{0},$$

where

$$\mathbf{P}_0=\begin{Vmatrix}1 & 0\\ 0 & \omega_1^2\end{Vmatrix},\quad \mathbf{P}_1(t)=-\left(\frac{\partial\mathbf{g}}{\partial\mathbf{v}}\right)_0=\begin{Vmatrix}4\beta_1^2\sin 2t & 0\\ 0 & 0\end{Vmatrix},$$

$$\mathbf{Q}_1(t)=-\left(\frac{\partial\mathbf{g}}{\partial\dot{\mathbf{v}}}\right)_0=(-1+2\beta_1^2-2\beta_1^2\cos 2t)\,\mathbf{I}_2.$$

Thus,

$$\mathbf{P}_1^{(2)}=\overline{\mathbf{P}}_1^{(-2)}=\begin{Vmatrix}-2i\beta_1^2 & 0\\ 0 & 0\end{Vmatrix},$$

$$\mathbf{Q}_1^{(0)}=(-1+2\beta_1^2)\,\mathbf{I}_2,\quad \mathbf{Q}_1^{(2)}=\mathbf{Q}_1^{(-2)}=-\beta_1^2\mathbf{I}_2,\quad \mathbf{I}_2=\begin{Vmatrix}1 & 0\\ 0 & 1\end{Vmatrix},$$

and all the other Fourier coefficients of matrix-functions $\mathbf{P}_1(t)$ and $\mathbf{Q}_1(t)$ vanish.

The roots of equations (5.3) are $\omega_{-1}=-1$, $\omega_1=1$, $\omega_{-2}=-\omega_2$, and ω_2. The eigenvectors $\mathbf{a}_1$ and $\mathbf{a}_2$ of the matrix $\mathbf{P}_0$ normalized by conditions (5.4) are

$$\mathbf{a}_1=\frac{1}{\sqrt{2}}\mathbf{e}_1,\quad \mathbf{a}_2=\frac{1}{\sqrt{2\omega_2}}\mathbf{e}_2.$$

There is a class of two roots $\{\omega_{-1},\ \omega_1\}$ that are congruent modulo $2\pi/T=1$, and all the other roots are by assumption $\left(\omega_2\neq\frac{1}{2}m\right)$ congruent neither to these two nor to one another, forming two classes each containing one root: $\{\omega_{-2}\}$ and $\{\omega_2\}$. We begin with the class of two roots, defining the integers m_{-1} and m_1 from the expressions

$$\omega_{-1}=0+(-1)\cdot 1,\quad \omega_1=0+1\cdot 1,$$

so that $m_{-1}=-1$ and $m_1=1$. We compute σ_{jh} from (6.1)

$$\sigma_{-1-1}=\overline{\sigma}_{11}=\frac{i}{2}(2\beta_1^2-1),\quad \sigma_{-11}=\overline{\sigma}_{1-1}=\frac{1}{2}\beta_1^2 i$$

and set up equation (6.2)

$$\begin{vmatrix}\sigma_{-1-1}+i\chi & \sigma_{-11}\\ \sigma_{1-1} & \sigma_{11}-i\chi\end{vmatrix}=\left[\chi+\frac{1}{2}(-1+2\beta_1^2)\right]^2-\frac{1}{4}\beta_1^4=0.$$

Its roots are

$$\chi_1=\frac{1}{2}(1-\beta_1^2),\quad \chi_2=\frac{1}{2}(1-3\beta_1^2).$$

If $\beta_1 = \beta_1^{(0)} > 1$, χ_1 and χ_2 are negative. If $\beta_1 = \beta_1^* < 0$ or $\beta_1 = \beta_1^{**} < 0$ (this is possible only if $q > -4\sqrt{3}/9$), χ_1 is positive, since by (7.4)

$$1-\beta_1^{*2} = \frac{1}{2}\frac{q}{\beta_1^*} > 0, \quad 1-\beta_1^{**2} = \frac{1}{2}\frac{q}{\beta_1^{**}} > 0.$$

By virtue of the conclusion of Subsection 2.6, the periodic solution (1.2) with the generating amplitudes $-2\beta_1^*$ and $-2\beta_1^{**}$ (see (7.5)) is unstable. To determine the stability of the periodic solution (1.2) with the generating amplitude $2\beta_1^{(0)}$, we must refer to the other two classes $\{\omega_{-2}\}$ and $\{\omega_2\}$. We evaluate ϑ_2 and ϑ_{-2} from (5.5′)

$$\vartheta_2 = -(\mathbf{Q}_1^{(0)}\mathbf{a}_2, \mathbf{a}_2) = \frac{1-2\beta_1^2}{2\omega_2}, \quad \vartheta_{-2} = \vartheta_2.$$

If $\beta_1 = \beta_1^{(0)} > 1$, ϑ_2 is negative. Thus the periodic solution (1.2) with the generating amplitude $2\beta_1^{(0)}$ is asymptotically stable.

Another example is analyzed in the following section.

§ 3. Forced Vibrations in Centrifuges Used for Spinning

3.1. Statement of the problem and equations of motion. The machinery used in manufacturing viscose rayon includes centrifuges for pot spinning. The machine is a combination of an electrically driven shaft and a centrifugal pot installed on the end of a vertical flexible cantilever spindle. The working rotation rate of spindles in modern centrifuges used for spinning reaches 9000 rpm. This angular velocity is substantially greater than the threshold at which the elastic properties of the spindle become significant. At a certain rotation rate, the inertial forces and moments of a rotating spindle are balanced by the elastic restoring forces and moments caused by deformation of the spindle carrying the centrifugal pot. In the accepted nomenclature, a shaft is called flexible if the fundamental frequency of transverse vibrations of the shaft is below its rotation rate. An important advantage of flexible-shaft centrifuges is their self-centring behaviour. Both the static and dynamic instabilities of the rotor tend to cancel out at the indicated rotation rates.

Under certain conditions, however, unstable modes of operation may appear in machines with flexible shafts. In addition to pure forced vibrations caused by rotor imbalance, self-excited vibrations are also possible; the frequencies of these undamped oscillations are close to the natural frequencies of the linearized system of differential equations describing the motion of the device in question.

One of the reasons for the instability of pure forced vibrations at above-critical frequencies may be internal friction in the material from which the flexible shaft is made ([104], pp. 108-113). According

to a well-known theoretical principle, forces of external resistance shift the stability limit toward higher frequencies [50, 51]. Therefore, it is of interest to find the relationships between the parameters of internal friction and external resistance that result in asymptotic stability of forced vibrations of the centrifuge.

To derive the differential equations describing the centrifuge's motion we use the approach suggested by Koritysskii ([96], p. 383). The variables introduced for the calculations in the case of the ЭВ-3М centrifuge are shown in Fig. 1.

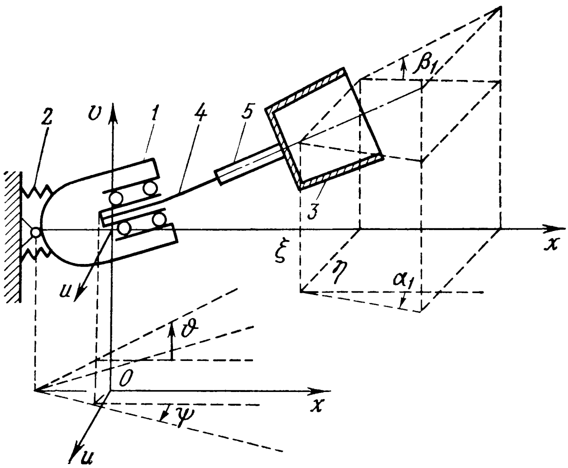

FIG. 1

The figure also shows the coordinates of the centrifugal pot with respect to the fixed reference frame xuv. In actual configurations the x axis is directed upward. According to the accepted dynamic model and with certain natural assumptions made by many authors [96, 104], the centrifuge vibrations can be described by a quasilinear nonautonomous system of differential equations with constant coefficients on the leading terms. In complex form, the system takes the form

$$m\ddot{u}_1 + c_{11}u_1 - c_{12}u_2 - c_{13}u_3 = me\nu^2 \exp(i\nu t)$$

$$- \mu \{m [\dot{u}_1 + ie\nu \exp(i\nu t)] (\varkappa_0 + \varkappa_1 \mid u_1 \mid^2)$$

$$+ hc_{11}(\dot{u}_1 - i\nu u_1) - hc_{12}(\dot{u}_2 - i\nu u_2) - hc_{13}(\dot{u}_3 - i\nu u_3)\},$$

$$K_1\ddot{u}_2 - i\nu K_0\dot{u}_2 - c_{21}u_1 + c_{22}u_2 + c_{23}u_3$$

$$= \mu \left[\left(\frac{1}{2} i\nu K_0\dot{u}_2 + c_{21}u_1 - c_{22}u_2 - c_{23}u_3 \right) \mid u_2 \mid^2 \right.$$

$$-(i\nu K_0 u_2 - 2K_1\dot{u}_2)\,\mathrm{Re}\,(\bar{u}_2\dot{u}_2) + hc_{21}(\dot{u}_1 - i\nu u_1)$$
$$-hc_{22}(\dot{u}_2 - i\nu u_2) - hc_{23}(\dot{u}_3 - i\nu u_3)\Big],$$
$$A\ddot{u}_3 - c_{31}u_1 + c_{32}u_2 + (c^0_{33} + f)\,u_3$$
$$= \mu\,[hc_{31}(\dot{u}_1 - i\nu u_1) - hc_{32}(\dot{u}_2 - i\nu u_2)$$
$$-hc^0_{33}(\dot{u}_3 - i\nu u_3) - A\varkappa_2\dot{u}_3]. \qquad (1.1)$$

The meaning of the complex coordinates u_j $(j = 1, 2, 3)$ is easily seen in Fig. 1: $u_1 = \eta + i\xi$, where η and ξ are the coordinates of the centroid of the rotor's rigid element *5* and the centrifugal pot *3*; $u_2 = \alpha_1 + i\beta_1$, where α_1 and β_1 are the Résal angles characterizing the direction of the tangent to the elastic axis of the vertical flexible cantilever shaft *4* at the point where the rigid portion of the rotor is mounted; $u_3 = \psi + i\vartheta$, where ψ and ϑ are the deviation angles of the axis of the spindle housing with respect to the fixed x axis.

The constant coefficients $c_{jk} = c_{kj}$ and c^0_{33} $(j, k = 1, 2, 3)$ are functions of design parameters of the mechanical system. The stiffness of the vibration absorbers *2* relative to angular displacements of the spindle housing *1* is given by the coefficient f. The linear eccentricity e of the rotor is assumed small in comparison with the leading terms of the differential equations. The remaining quantities in equations (1.1) are: K_0 and K_1 are, respectively, the polar and equatorial central moments of inertia of the rotor; A is the equatorial moment of inertia of the system with respect to its fixed point; m is the rotor mass; $\varkappa_0$, $\varkappa_1$, and $\varkappa_2$ are the coefficients of external dissipative forces; and h is the coefficient of internal friction for the material of the flexible shaft [50, 96].

Finally, we note that the parameter μ is introduced in the differential equations of motion (1.1) in order to single out the terms small in comparison with those in the left-hand sides of these equations.

3.2. Determination of a periodic solution. Experimental data on the functioning of centrifuges have demonstrated that pure forced vibrations are caused by rotor imbalance. These vibrations correspond to a periodic solution of the initial differential equations (1.1), which we shall seek by the Poincaré method. The initial system of three second-order differential equations (1.1) in the complex-variable functions u_j $(j = 1, 2, 3)$ is first recast in the form of a single real-variable vector equation (2, 1.1):

$$\mathbf{M}\frac{d^2\mathbf{v}}{dt^2} + \mathbf{Q}_0\frac{d\mathbf{v}}{dt} + \mathbf{P}_0\mathbf{v} = \mathbf{f}(t) + \mu\mathbf{g}(t, \mathbf{v}, \dot{\mathbf{v}}). \qquad (2.1)$$

Here we introduce the following notation: the vectors

$$\mathbf{v}=\begin{Vmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{Vmatrix}, \quad \mathbf{f}(t)=me\nu^2\begin{Vmatrix} \cos \nu t \\ \sin \nu t \\ 0 \\ 0 \\ 0 \\ 0 \end{Vmatrix}, \quad \mu\mathbf{g}(t,\ \mathbf{v},\ \dot{\mathbf{v}})=\mu\begin{Vmatrix} g_1 \\ g_2 \\ g_3 \\ g_4 \\ g_5 \\ g_6 \end{Vmatrix},$$

where $x_1 = \eta$, $x_2 = \xi$, $x_3 = \alpha_1$, $x_4 = \beta_1$, $x_5 = \psi$, $x_6 = \vartheta$, and the functions μg_j $(j = 1, \ldots, 6)$ are found as a result of splitting the corresponding complex equations of system (1.1); and the matrices

$$\mathbf{M} = \operatorname{diag}(m,\ n,\ K_1,\ K_1,\ A,\ A), \quad \mathbf{Q}_0 = \| q_{jh} \|_1^6,$$

where $q_{34} = K_0\nu$, $q_{43} = -K_0\nu$, and the remaining elements of the matrix $\mathbf{Q}_0$ are zero.

Splitting the initial system of complex equations (1.1) yields a matrix $\mathbf{P}_0$, which for the system in question is symmetric and positive-definite. The elements of this 6×6 matrix are the coefficients $c_{jk} = c_{kj}$ and $c_{33}^0 + f$ $(j,\ k = 1,\ 2,\ 3)$. The vector $\mathbf{v}$ thus gives the coordinates of the mechanism, and the vector-functions $\mu\mathbf{g}(t,\ \mathbf{v},\ \dot{\mathbf{v}})$ are periodic in time t with period $T = 2\pi/\nu$.

Let us seek the T-periodic solution of equation (2.1) in the form of a series (2, 1.2) in integer powers of the small parameter μ, keeping only the first correction to the generating solution

$$\mathbf{v}(t,\ \mu) = \mathbf{v}^{(0)}(t) + \mu\mathbf{v}^{(1)}(t) + \ldots \tag{2.2}$$

By using the standard procedure (see Subsection 2.1), we arrive at vector differential equations (2. 1.3) for the generating solution, and (2, 1.4) for the first correction. The characteristic equation (2, 1.6) for the homogeneous system (2, 1.5) is

$$[\det(\mathbf{P}_0 - \lambda\mathbf{I}_6)]^2 + \{[(c_{11} + m\lambda^2)(c_{33}^0 + f + A\lambda^2) - c_{13}^2] K_0\nu\}^2 \lambda^2 = 0.$$

Obviously, this equation has the pure imaginary roots $\lambda_j = i\omega_j$ $(j = \mp 1, \ldots, \mp 6)$. We assume that no one of these roots is of the form $ip\nu$ $(p = 0, \mp 1, \mp 2, \ldots;\ i = \sqrt{-1})$. In this case both the equation for the generating solution and each subsequent equation as well will have a unique T-periodic solution. To find these solutions it is expedient to return to the complex notation of the initial equations. The generating solution then takes the form

$$x_{2k-1}^{(0)} + ix_{2k}^{(0)} = me\nu^2 D^{-1}(\nu)\, D_{1k}(\nu) \exp(i\nu t) \quad (k = 1,\ 2,\ 3).$$

We recall that $x_j^{(0)}$ $(j = 1, \ldots, 6)$ are the components of the vector $\mathbf{v}^{(0)}$ of the generating solution, and $D_{1k}(\nu)$ are algebraic comple-

ments of the corresponding elements in the fundamental determinant

$$D(\nu) \equiv \det(-\nu^2\widetilde{\mathbf{M}} + i\nu\widetilde{\mathbf{Q}}_0 + \widetilde{\mathbf{P}}_0).$$

Here

$$\widetilde{\mathbf{M}} = \operatorname{diag}\ (m,\ K_1,\ A),$$
$$\widetilde{\mathbf{Q}}_0 = \operatorname{diag}\ (0,\ -i\nu K_0,\ 0),$$
$$\widetilde{\mathbf{P}}_0 = \begin{Vmatrix} c_{11} & -c_{12} & -c_{13} \\ -c_{12} & c_{22} & c_{23} \\ -c_{13} & c_{23} & c_{33}^0 + f \end{Vmatrix}.$$

First corrections are found from

$$\mu\left(x_{2k-1}^{(1)} + ix_{2k}^{(1)}\right) = D^{-1}(\nu) \sum_{j=1}^{3} r_j D_{jk}(\nu) \quad (k = 1,\ 2,\ 3).$$

The following symbols were introduced in the last expression: $x_j^{(1)}$ $(j = 1, \ldots, 6)$ are the components of the vector $\mathbf{v}^{(1)}$,

$$r_k = g_{2k-1}(t,\ \mathbf{v}^{(0)},\ \dot{\mathbf{v}}^{(0)}) + ig_{2k}(t,\ \mathbf{v}^{(0)},\ \dot{\mathbf{v}}^{(0)}) \quad (k = 1, 2, 3).$$

Therefore,

$$r_1 = i\nu me\left[\varkappa_0 + \varkappa_1\left(\frac{me\nu^2 D_{11}(\nu)}{D(\nu)}\right)^2\right]\left(1 + \frac{m\nu^2 D_{11}(\nu)}{D(\nu)}\right)\exp(i\nu t),$$
$$r_2 = -\left(\frac{me\nu^2 D_{12}(\nu)}{D(\nu)}\right)^3 K_1\nu^2\left(1 - \frac{K_0}{2K_1}\right)\exp(i\nu t),$$
$$r_3 = -i\nu A\varkappa_2 \frac{me\nu^2 D_{13}(\nu)}{D(\nu)} \exp(i\nu t).$$

The higher-order approximations are found in the same manner.

3.3. Stability analysis. Physically, as we know, only stable motions are realized. Therefore, the periodic solution found in the preceding subsection must be analyzed for stability, which should at the same time establish the domains of existence of the periodic motions of the centrifuge. For this analysis, we need variational equations, assuming the periodic motion already obtained to be the unperturbed motion.

We denote by the vector $\mathbf{y}$ a small deviation from the unperturbed motion; the perturbed motion is then

$$\mathbf{v} = \mathbf{v}(t,\ \mu) + \mathbf{y}.$$

By substituting the perturbed motion into system (2.1), we recast the variational equations in vector form (2, 4.2) and drop the terms containing μ to the second and higher powers

$$\mathbf{M}\frac{d^2\mathbf{y}}{dt^2} + [\mathbf{Q}_0 + \mu\mathbf{Q}_1(t) + \ldots]\frac{d\mathbf{y}}{dt} + [\mathbf{P}_0 + \mu\mathbf{P}_1(t) + \ldots]\mathbf{y} = \mathbf{0}. \quad (3.1)$$

As was noted above, all the roots of the characteristic equation in the case under investigation are pure imaginary, and all the multipliers of system (3.1) for $\mu = 0$ corresponding to period $T = 2\pi/\nu$ are distinct. The characteristic exponents of system (3.1) are found from (2, 5.3)

$$\alpha_j(\mu) = i\omega_j - i\sigma_{jj}\,\mathrm{sign}\, j\mu + O(\mu^2) \quad (j = \mp 1, \ldots, \mp 6),$$

where $\sigma_{jj} = -([\mathbf{P}_1^{(0)} + i\omega_j\mathbf{Q}_1^{(0)}]\,\mathbf{a}_j, \mathbf{a}_j)$ and $\mathbf{a}_j$ are normalized eigenvectors

$$(-\omega_j^2\mathbf{M} + i\omega_j\mathbf{Q}_0 + \mathbf{P}_0)\,\mathbf{a}_j = \mathbf{0},$$

$$([2\omega_j\mathbf{M} - i\mathbf{Q}_0]\,\mathbf{a}_j, \mathbf{a}_j) = \mathrm{sign}\, j \quad (j = \mp 1, \ldots, \mp 6).$$

The following quantities can be taken as components of the eigenvectors

$$a_j^{(1)} = \frac{D_{21}(\omega_j)}{D_{22}(\omega_j)}c_j, \quad a_j^{(2)} = \frac{D_{21}(\omega_j)}{D_{22}(\omega_j)}, \quad a_j^{(3)} = c_j,$$

$$a_j^{(4)} = c_j, \quad a_j^{(5)} = \frac{D_{23}(\omega_j)}{D_{22}(\omega_j)}c_j, \quad a_j^{(6)} = \frac{D_{23}(\omega_j)}{D_{22}(\omega_j)}$$

$$(j = \mp 1, \ldots, \mp 6),$$

where

$$c_j^2 = \frac{1}{4}\omega_j^{-1}\left[m\frac{D_{21}(\omega_j)^2}{D_{22}(\omega_j)^2} + K_1\left(1 - \frac{K_0\nu}{2K_1\omega_j}\right) + A\frac{D_{23}(\omega_j)^2}{D_{22}(\omega_j)^2}\right]^{-1}\mathrm{sign}\, j.$$

Normalization becomes unimportant if stability is being analyzed for the case of distinct multipliers.

The products $(\mathbf{P}_1^{(0)}\mathbf{a}_j, \mathbf{a}_j)$ being real $(j = \mp 1, \ldots, \mp 6)$, conditions (2.2) for asymptotic stability of the motion take the form of (2, 5.5′)

$$\mathrm{Re}\,(\mathbf{Q}_1^{(0)}\mathbf{a}_j, \mathbf{a}_j) > 0 \quad (j = 1, \ldots, 6).$$

The matrix $\mu\mathbf{Q}_1^{(0)}$ for system (2.1) is

$$\left\|\begin{matrix} M + hc_{11} & 0 & -hc_{12} & 0 & -hc_{13} & 0 \\ 0 & M + hc_{11} & 0 & -hc_{12} & 0 & -hc_{13} \\ -hc_{12} & 0 & hc_{22} & K_1\nu f_2^2 & hc_{23} & 0 \\ 0 & -hc_{12} & -K_1\nu f_2^2 & hc_{22} & 0 & hc_{23} \\ -hc_{13} & 0 & hc_{23} & 0 & hc_{33}^0 + A\varkappa_2 & 0 \\ 0 & -hc_{13} & 0 & hc_{23} & 0 & hc_{33}^0 + A\varkappa_2 \end{matrix}\right\|,$$

where

$$M = m(\varkappa_0 + \varkappa_1 f_1^2), \quad f_k = me\nu^2 D^{-1}(\nu)\, D_{1k}(\nu) \quad (k = 1, 2).$$

In full form, the conditions for asymptotic stability of pure forced vibrations of the centrifuge used for spinning are

$$m(\varkappa_0 + \varkappa_1 f_1^2) D_{21}(\omega_j)^2 + A\varkappa_2 D_{23}(\omega_j)^2 + hm\omega_j^2 D_{21}(\omega_j)^2 - h\omega_j (K_0\nu - K_1\omega_j) D_{22}(\omega_j)^2 - h(f - A\omega_j^2) D_{23}(\omega_j)^2 > 0 \quad (j = 1, \ldots, 6).$$

Here $D_{2k}(\omega_j)$ are algebraic complements of the corresponding elements of the fundamental determinant of system (2.1).

An analysis of the conditions for asymptotic stability makes it possible to conclude that forces of internal friction in devices similar to the centrifuge used for spinning discussed above cannot disrupt the asymptotic stability of forced vibrations caused by rotor imbalance.

CHAPTER II

OSCILLATORY CHAINS

In the next two sections we treat plane oscillatory chains. Section 1 [371d] deals with completely elastic free chains; if we consider a mechanical system, this is a system of point masses linked by weightless springs. Section 4 [371c] discusses a case when one of these conditions is violated, namely, some of the springs are replaced by weightless rods, and in one of the examples the motion of some of the point masses is constrained by guides; therefore, the oscillatory chain is no longer completely elastic and free.

At first glance, these are special cases of mechanical vibrations. Such a view, however, is incorrect. Even in some problems of mechanics a rod can be treated as an oscillatory chain, and it is not always easy to decide whether a continuous or a discrete model gives a better description of reality. Analogies become even more profound if one considers electrodynamic systems. We refer the reader to Mandelshtam's monograph [112] (Part I, Lecture 29; Part II, Lecture 12). In Section 3 of Part II we discuss the relationship between oscillations of particles in cyclic accelerators and those of spring-loaded pendulums (mass-spring systems); this is only one of numerous possible examples.

§ 1. Completely Elastic Free Oscillatory Chains

1.1. Definition of an oscillatory chain. We consider a mechanical system with holonomic constraints that do not depend explicitly on time. Let $q_1, \ldots, q_n$ be the Lagrangian coordinates of the system and $\dot{q}_1, \ldots, \dot{q}_n$ be the corresponding generalized velocities. We assume that a generalized force corresponding to the coordinate q_ν can be given in the form

$$Q_\nu(q_1, \ldots, q_n) - R_\nu(\dot{q}_1, \ldots, \dot{q}_n) \quad (\nu = 1, \ldots, n).$$

Here Q_ν and R_ν are continuous and differentiable functions of their arguments in the respective domains of definition. We assume that for any virtual displacement (which in this particular case coincides with a real displacement) the work of resistance forces is negative

$$-\sum_{\nu=1}^{n} R_\nu(\dot{q}_1, \ldots, \dot{q}_n)\,\dot{q}_\nu < 0. \qquad (1.1)$$

It follows from this inequality and from continuity that

$$R_\nu(0, \ldots, 0) = 0 \quad (\nu = 1, \ldots, n).$$

In the simplest nonlinear case, when $R_\nu = f(\dot{q}_\nu)$ $(\nu = 1, \ldots, n)$, condition (1.1) signifies that $\alpha f(\alpha) > 0$ $(\alpha \neq 0)$, and the continuity requirement implies, among other things, that $f(0) = 0$. In the linear case, (1.1) signifies complete dissipation.

The constraints being explicitly independent of time, the kinetic energy K of the system is a quadratic form of generalized velocities with coefficients dependent only on the Lagrangian coordinates

$$K = \frac{1}{2} \sum_{i,j=1}^{n} a_{ij}(q_1, \ldots, q_n)\dot{q}_i\dot{q}_j \quad (a_{ji} = a_{ij};\ i, j = 1, \ldots, n).$$

The equations of motion of the second kind in the Lagrange form can be written as

$$\sum_{i=1}^{n} a_{\nu i}\ddot{q}_i + \sum_{i,j=1}^{n} \left(\frac{\partial a_{\nu i}}{\partial q_j} - \frac{1}{2}\frac{\partial a_{ij}}{\partial q_\nu}\right)\dot{q}_i\dot{q}_j = Q_\nu - R_\nu \tag{1.2}$$

$$(\nu = 1, \ldots, n).$$

We wish to analyze the stability of unperturbed motion in the sense of Lyapunov

$$q_\nu = q_{\nu 0}(t), \quad \dot{q}_\nu = \dot{q}_{\nu 0}(t) \quad (\nu = 1, \ldots, n) \tag{1.3}$$

with respect to the variables $q_1, \ldots, q_r, \dot{q}_1, \ldots, \dot{q}_r$ $(r \leqslant n)$. Coordinates and velocities of the perturbed motion are denoted by

$$q_\nu = q_{\nu 0}(t) + \varkappa_\nu, \quad \dot{q}_\nu = \dot{q}_{\nu 0}(t) + \dot{\varkappa}_\nu \quad (\nu = 1, \ldots, n).$$

Differential equations for the first approximation of the perturbed motion (variational equations) can be given as

$$\sum_{i=1}^{n} \left[(a_{\nu i})_0 \frac{d^2\varkappa_i}{dt^2} + b_{\nu i}(t)\frac{d\varkappa_i}{dt} + c_{\nu i}(t)\varkappa_i\right] = 0 \quad (\nu = 1, \ldots, n), \tag{1.4}$$

where

$$b_{\nu i}(t) = \sum_{j=1}^{n} \left[\left(\frac{\partial a_{\nu j}}{\partial q_i}\right)_0 + \left(\frac{\partial a_{\nu i}}{\partial q_j}\right)_0 - \left(\frac{\partial a_{ij}}{\partial q_\nu}\right)_0\right]\dot{q}_{j0}(t) + \left(\frac{\partial R_\nu}{\partial \dot{q}_i}\right)_0, \tag{1.5}$$

$$c_{\nu i}(t) = \sum_{i=1}^{n} \left\{\left(\frac{\partial a_{\nu j}}{\partial q_i}\right)_0 \ddot{q}_{j0}(t) + \sum_{k=1}^{n} \left[\left(\frac{\partial^2 a_{\nu k}}{\partial q_i\,\partial q_j}\right)_0 - \frac{1}{2}\left(\frac{\partial^2 a_{jk}}{\partial q_\nu\,\partial q_i}\right)_0\right]\dot{q}_{j0}(t)\,\dot{q}_{k0}(t)\right\} - \left(\frac{\partial Q_\nu}{\partial q_i}\right)_0 \quad (\nu, i = 1, \ldots, n), \tag{1.6}$$

and the zero subscript on $a_{\nu i}$ and on the partial derivatives indicates evaluation at

$$q_{10}(t), \ldots, q_{n0}(t), \quad \dot{q}_{10}(t), \ldots, \dot{q}_{n0}(t).$$

We say that the initial mechanical system is an "oscillatory chain" with respect to the unperturbed motion (1.3) if it is possible to choose Lagrangian coordinates in which the coefficients $(a_{\nu i})_0$, $b_{\nu i}(t)$, and $c_{\nu i}(t)$ are such that for some natural $m < n$

$$(a_{\nu i})_0 = 0 \tag{1.7}$$

$$(\nu = 1, \ldots, m, \ i = m+1, \ldots, n;$$
$$\nu = m+1, \ldots, n, \ i = 1, \ldots, m);$$

$$b_{\nu i}(t) = \left(\frac{\partial R_\nu}{\partial \dot{q}_i}\right)_0 \quad (\nu, \ i = 1, \ldots, n), \tag{1.8}$$

$$\left(\frac{\partial R_\nu}{\partial \dot{q}_i}\right)_0 = 0 \tag{1.9}$$

$$[(\nu = 1, \ldots, m, \ i = m+1, \ldots, n;$$
$$\nu = m+1, \ldots, n, \ i = 1, \ldots, m);$$

$$c_{\nu i}(t) = 0 \tag{1.10}$$

$$[(\nu = 1, \ldots, m, \ i = m+1, \ldots, n;$$
$$\nu = m+1, \ldots, n, \ i = 1, \ldots, m)$$

for any $t \geqslant t_0$. Conditions (1.7)-(1.10) state that the matrix-functions of the coefficients of system (1.4) are of the form

$$\begin{pmatrix} \|(a_{\nu i})_0\|_1^m & 0 \\ 0 & \|(a_{\nu i})_0\|_{m+1}^n \end{pmatrix}, \quad \begin{pmatrix} \|c_{\nu i}(t)\|_1^m & 0 \\ 0 & \|c_{\nu i}(t)\|_{m+1}^n \end{pmatrix},$$

$$\begin{pmatrix} \left\|\left(\frac{\partial R_\nu}{\partial \dot{q}_i}\right)_0\right\|_1^m & 0 \\ 0 & \left\|\left(\frac{\partial R_\nu}{\partial \dot{q}_i}\right)_0\right\|_{m+1}^n \end{pmatrix}.$$

If conditions (1.7)-(1.10) are satisfied, the variational equations (1.4) form two groups of m and $n-m$ equations

$$\sum_{i=1}^{m} \left[(a_{\nu i})_0 \frac{d^2 \varkappa_i}{dt^2} + \left(\frac{\partial R_\nu}{\partial \dot{q}_i}\right)_0 \frac{d\varkappa_i}{dt} + c_{\nu i}(t)\,\varkappa_i\right] = 0 \tag{1.11}$$

$$(\nu = 1, \ldots, m),$$

$$\sum_{i=m+1}^{n} \left[(a_{\nu i})_0 \frac{d^2 \varkappa_i}{dt^2} + \left(\frac{\partial R_\nu}{\partial \dot{q}_i}\right)_0 \frac{d\varkappa_i}{dt} + c_{\nu i}(t)\,\varkappa_i\right] = 0 \tag{1.12}$$

$$(\nu = m+1, \ldots, n).$$

1.2. Determination of equilibrium positions. The simplest example of an "oscillatory chain" is a *completely elastic free oscillatory chain* with respect to vertical vibrations (this means that vertical vibrations are taken for unperturbed motion). Figure 2 shows a system of N point masses $m_1, \ldots, m_N$ connected in series by N zero-mass springs with force constants $c_1, \ldots, c_N$ and of unstressed lengths $l_1, \ldots, l_N$. The first spring is fixed at point O, and each subsequent spring is attached to a weightless hinge whose axis is perpendicular to the vertical plane Oxy, which determines the motion in a plane. Thus, only N trivial constraints are imposed on the system: $z_1 = 0, \ldots, z_N = 0$; the Cartesian coordinates of the point masses $m_1, \ldots, m_N$ are taken for the $2N$ Lagrangian coordinates. For this simplest case the kinetic energy is

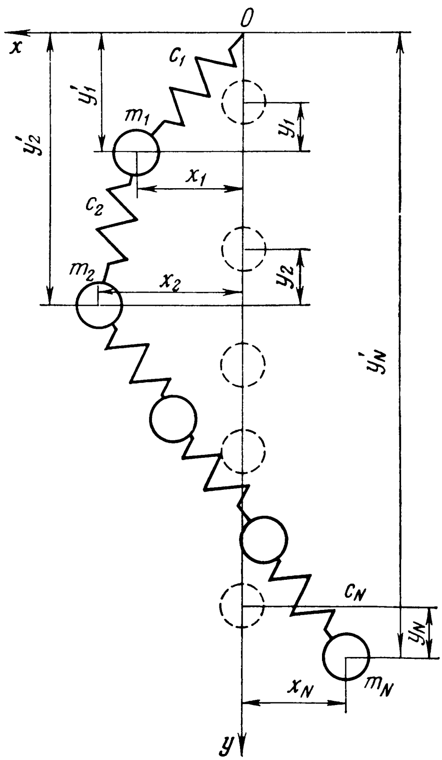

FIG. 2

$$K = \frac{1}{2} \sum_{k=1}^{N} m_k (\dot{x}_k^2 + \dot{y}_k'^2),$$

that is, $a_{ij} = m_i \delta_{ij}$ (δ_{ij} is the Kronecker delta; $i, j = 1, \ldots, N$). Let us calculate the potential energy $\Pi(x_1, \ldots, x_N, y_1', \ldots, y_N')$ of the linear elastic forces of the springs and of the forces of gravity

$$\Pi = \sum_{k=1}^{N} \Big\{ -g m_k y_k' + \frac{1}{2} c_k [(x_k - x_{k-1})^2 + (y_k' - y_{k-1}')^2$$

$$- 2 l_k \sqrt{(x_k - x_{k-1})^2 + (y_k' - y_{k-1}')^2}] \Big\}, \qquad (2.1)$$

assuming $x_0 = y_0' = 0$ and taking only the arithmetic value of the square root. We start by finding the equilibrium positions of

the system and for this consider the system of equations

$$\frac{\partial \Pi}{\partial x_k} = -c_k x_{k-1} + (c_k + c_{k+1}) x_k - c_{k+1} x_{k+1}$$
$$- c_k l_k \left[(x_k - x_{k-1})^2 + (y'_k - y'_{k-1})^2\right]^{-1/2} (x_k - x_{k-1})$$
$$+ c_{k+1} l_{k+1} \left[(x_{k+1} - x_k)^2 + (y'_{k+1} - y'_k)^2\right]^{-1/2} (x_{k+1} - x_k) = 0,$$
$$\frac{\partial \Pi}{\partial y'_k} = -m_k g - c_k y'_{k-1} + (c_k + c_{k+1}) y'_k - c_{k+1} y'_{k+1}$$
$$- c_k l_k \left[(x_k - x_{k-1})^2 + (y'_k - y'_{k-1})^2\right]^{-1/2} (y'_k - y'_{k-1})$$
$$+ c_{k+1} l_{k+1} \left[(x_{k+1} - x_k)^2 + (y'_{k+1} - y'_k)^2\right]^{-1/2} (y'_{k+1} - y'_k) = 0 \quad (2.2)$$
$$(k = 1, \ldots, N),$$

in which we assume $c_{N+1} = l_{N+1} = 0$. A solution of this system (for the lower equilibrium position) is

$$x_k = 0, \quad y'_k = (l_1 + \lambda_1) + \ldots + (l_k + \lambda_k) \qquad (2.3)$$
$$(k = 1, \ldots, N),$$

where λ_j is the static elongation of the jth spring,

$$\lambda_j = (m_j + m_{j+1} + \ldots + m_N)\, g/c_j \quad (j = 1, \ldots, N).$$

Let us prove that this equilibrium position is isolated. Indeed, the Jacobian of system (2.2) evaluated at x_k and y'_k from (2.3) is $D = D_1 D_2$, where D_1 and D_2 are the determinants of the Jacobian matrices of order N

$$D_1 = \begin{vmatrix} \frac{c_1\lambda_1}{l_1+\lambda_1} + \frac{c_2\lambda_2}{l_2+\lambda_2} & -\frac{c_2\lambda_2}{l_2+\lambda_2} & 0 & \ldots & 0 \\ -\frac{c_2\lambda_2}{l_2+\lambda_2} & \frac{c_2\lambda_2}{l_2+\lambda_2} + \frac{c_3\lambda_3}{l_3+\lambda_3} & -\frac{c_3\lambda_3}{l_3+\lambda_3} & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & \frac{c_N\lambda_N}{l_N+\lambda_N} \end{vmatrix},$$

and D_2 is obtained from D_1 when $l_k = 0, \lambda_k = 1$ $(k = 1, \ldots, N)$. The formula for the determinant of a Jacobian matrix yields

$$D = \prod_{k=1}^{N} \frac{c_k^2 \lambda_k}{l_k + \lambda_k} > 0,$$

which was to be demonstrated.

Let us introduce new variables y_k defined as the displacements of the kth point along the vertical axis from the lower equilibrium position

$$y_k = y'_k - (l_1 + \lambda_1) - \ldots - (l_k + \lambda_k) \quad (k = 1, \ldots, N). \quad (2.4)$$

In the lower equilibrium position, $x_1 = \ldots = x_N = y_1 = \ldots$ $\ldots = y_N = 0$. In order to find equilibrium positions not coinciding with the lower one, we recast system (2.2) in the form

$$\begin{aligned}
&-c_k (x_k - x_{k-1}) \\
&\quad \times \{l_k [(x_k - x_{k-1})^2 + (l_k + \lambda_k + y_k - y_{k-1})^2]^{-1/2} - 1\} \\
&\qquad + c_{k+1} (x_{k+1} - x_k) \{l_{k+1} [(x_{k+1} - x_k)^2 \\
&\qquad\qquad + (l_{k+1} + \lambda_{k+1} + y_{k+1} - y_k)^2]^{-1/2} - 1\} = 0, \\
&-c_k (l_k + \lambda_k + y_k - y_{k-1}) \\
&\quad \times \{l_k [(x_k - x_{k-1})^2 + (l_k + \lambda_k + y_k - y_{k-1})^2]^{-1/2} - 1\} \\
&\quad + c_{k+1} (l_{k+1} + \lambda_{k+1} + y_{k+1} - y_k) \{l_{k+1} [(x_{k+1} - x_k)^2 \\
&\qquad + (l_{k+1} + \lambda_{k+1} + y_{k+1} - y_k)^2]^{-1/2} - 1\} = m_k g. \qquad (2.5) \\
&\qquad\qquad (k = 1, \ldots, N).
\end{aligned}$$

The equations corresponding to $k = N$ are

$$\begin{aligned}
&c_N (x_N - x_{N-1}) \{1 - l_N [(x_N - x_{N-1})^2 \\
&\qquad + (l_N + \lambda_N + y_N - y_{N-1})^2]^{-1/2}\} = 0, \\
&c_N (l_N + \lambda_N + y_N - y_{N-1}) \{1 - l_N [(x_N - x_{N-1})^2 \\
&\qquad + (l_N + \lambda_N + y_N - y_{N-1})^2]^{-1/2}\} = m_N g.
\end{aligned}$$

The expression in braces is distinct from zero, otherwise the second equation would not be true. Therefore, the first equation yields $x_N = x_{N-1}$, and the second equation becomes

$$\begin{aligned}
&c_N (l_N + \lambda_N + y_N - y_{N-1}) \\
&\qquad \times \{1 - l_N \mid l_N + \lambda_N + y_N - y_N \mid^{-1}\} = m_N g.
\end{aligned}$$

This last equation always has the solution $y_N = y_{N-1}$, and also $y_N = y_{N-1} - 2l_N$ if $\lambda_N < l_N$. Let us consider the frequent case when the static elongation of each of the springs is less than the initial unstressed lengths

$$\lambda_k < l_k \quad (k = 1, \ldots, N). \qquad (2.6)$$

By treating equations (2.5) corresponding to $k = N - 1, N - 2, \ldots$ $\ldots, 1$ in a similar manner we determine 2^N equilibrium positions of the free oscillatory chain

$$x_1 = \ldots = x_N = 0,$$

$$y_1 = \begin{cases} 0 \\ -2l_1 \end{cases}, \quad y_2 = \begin{cases} y_1 \\ y_1 - 2l_2 \end{cases}, \quad \ldots, \quad y_N = \begin{cases} y_{N-1} \\ y_{N-1} - 2l_N \end{cases}. \qquad (2.7)$$

It should be stipulated that the planes in which each of the N point masses moves are distinct and vertical.

1.3. Asymptotic stability in the large of the lower equilibrium position for distinct resistance forces. The equations of motion (1.2) for a completely elastic free oscillatory chain are very simple

$$m_k \ddot{x}_k = -\frac{\partial \Pi}{\partial x_k} - R_k(\dot{x}_1, \ldots, \dot{x}_N, \dot{y}_1, \ldots, \dot{y}_N),$$

$$m_k \ddot{y}_k = -\frac{\partial \Pi}{\partial y_k} - R_{N+k}(\dot{x}_1, \ldots, \dot{x}_N, \dot{y}_1, \ldots, \dot{y}_N) \tag{3.1}$$

$$(k = 1, \ldots, N).$$

Let $\inf \Pi$ denote the lowest potential energy of the completely elastic free oscillatory chain on the *set of* $2^N - 1$ *equilibrium positions* distinct from the lower one. Define a closed domain G in the phase space $x_1, \ldots, x_N, y_1, \ldots, y_N, \dot{x}_1, \ldots, \dot{x}_N, \dot{y}_1, \ldots, \dot{y}_N$ by the inequality

$$K + \Pi \leqslant \inf \Pi.$$

Theorem. *If there exist resistance forces satisfying* (1.1), *the lower equilibrium position of a completely elastic free oscillatory chain is asymptotically stable for the initial displacements*

$$x_1^{(0)}, \ldots, x_N^{(0)}, \; y_1^{(0)}, \ldots, y_N^{(0)}, \; \dot{x}_1^{(0)}, \ldots, \dot{x}_N^{(0)}, \; \dot{y}_1^{(0)}, \ldots, \dot{y}_N^{(0)}$$

belonging to domain G. This means that $K^{(0)} + \Pi^{(0)}$ *satisfies the inequality*

$$K^{(0)} + \Pi^{(0)} < \inf \Pi, \tag{3.2}$$

where $K^{(0)}$ *and* $\Pi^{(0)}$ *are evaluated at* $x_1 = x_1^{(0)}, \ldots, \dot{y}_N = \dot{y}_N^{(0)}$.

Proof. We shall employ the total energy of the system

$$v = K + \Pi - \Pi(0, \ldots, 0) \tag{3.3}$$

as the function v of Krasovskii's theorem 14.1 [100]. We calculate the potential energy $\Pi(0, \ldots, 0)$ of the lower equilibrium position

$$\Pi(0, \ldots, 0) = -\sum_{k=1}^{N} \Big\{ m_k g \left[(l_1 + \lambda_1) + \ldots + (l_k + \lambda_k) \right] + \frac{1}{2} c_k (l_k^2 - \lambda_k^2) \Big\}.$$

We wish to prove that $\Pi - \Pi(0, \ldots, 0)$ is a positive-definite function, in the sense of Lyapunov, of $x_1, \ldots, x_N, y_1, \ldots, y_N$.

We transform $\Pi - \Pi(0, \ldots, 0)$ to

$$\Pi - \Pi(0, \ldots, 0) = \frac{1}{2} \sum_{k=1}^{N} c_k \left[(x_k - x_{k-1})^2 + (y_k - y_{k-1})^2 \right.$$
$$+ 2l_k (l_k + \lambda_k + y_k - y_{k-1})$$
$$\left. - 2l_k \sqrt{(x_k - x_{k-1})^2 + (l_k + \lambda_k + y_k - y_{k-1})^2}\right]$$

and establish the validity of the inequalities

$$(x_k - x_{k-1})^2 + (y_k - y_{k-1})^2 + 2l_k (l_k + \lambda_k + y_k - y_{k-1})$$
$$\geqslant 2l_k \sqrt{(x_k - x_{k-1})^2 + (l_k + \lambda_k + y_k - y_{k-1})^2} \tag{3.4}$$

($k = 1, \ldots, N$; $x_0 = y_0 = 0$). The left-hand side of these inequalities can be written as

$$(x_k - x_{k-1})^2 + (y_k - y_{k-1} + l_k)^2 + l_k (l_k + 2\lambda_k).$$

Obviously, it is positive. Raising inequalities (3.4) to the second power and transforming them, we arrive at

$$[(x_k - x_{k-1})^2 + (y_k - y_{k-1})^2 + 2l_k (y_k - y_{k-1})]^2$$
$$+ 4l_k\lambda_k [(x_k - x_{k-1})^2 + (y_k - y_{k-1})^2] \geqslant 0 \quad (k = 1, \ldots, N).$$

These inequalities are true, and equality holds only when $x_1 = \ldots = x_N = y_1 = \ldots = y_N = 0$. Hence, the total energy (3.3) of the system is a positive-definite function, in the sense of Lyapunov, of all the Lagrangian coordinates and velocities. According to equations (3.1), its derivative is

$$\frac{d}{dt}[K + \Pi - \Pi(0, \ldots, 0)] = -\sum_{k=1}^{N} (R_k \dot{x}_k + R_{N+k} \dot{y}_k) \leqslant 0.$$

By definition of the resistance forces, the equality sign in the last inequality is only possible in the equilibrium position. A motion initiated within G cannot leave it, while the equilibrium position in G is unique. The conditions of Krasovskii's theorem 14.1 are thus satisfied [100], proving the theorem in question.

Remark. Formulas can be found for the radius of a sphere or edge of a cube inscribed into the $4N$-dimensional domain G.

1.4. Variational equations for vertical oscillations of the system. In full form, equations (3.1) are

$$m_k \ddot{x}_k = -c_k (x_k - x_{k-1}) + c_{k+1} (x_{k+1} - x_k) + c_k l_k (x_k - x_{k-1})$$
$$\times [(x_k - x_{k-1})^2 + (l_k + \lambda_k + y_k - y_{k-1})^2]^{-1/2} - c_{k+1} l_{k+1} (x_{k+1} - x_k)$$
$$\times [(x_{k+1} - x_k)^2 + (l_{k+1} + \lambda_{k+1} + y_{k+1} - y_k)^2]^{-1/2}$$
$$- R_k (\dot{x}_1, \ldots, \dot{x}_N, \dot{y}_1, \ldots, \dot{y}_N),$$

$$
\begin{aligned}
m_k\ddot{y}_k = &-c_kl_k+c_{k+1}l_{k+1}-c_k(y_k-y_{k-1})\\
&+c_{k+1}(y_{k+1}-y_k)+c_kl_k(l_k+\lambda_k+y_k-y_{k-1})\\
&\times[(x_k-x_{k-1})^2+(l_k+\lambda_k+y_k-y_{k-1})^2]^{-1/2}\\
&-c_{k+1}l_{k+1}(l_{k+1}+\lambda_{k+1}+y_{k+1}-y_k)\\
&\times[(x_{k+1}-x_k)^2+(l_{k+1}+\lambda_{k+1}+y_{k+1}-y_k)^2]^{-1/2}\\
&\qquad\qquad -R_{N+k}(\dot{x}_1,\ldots,\dot{x}_N,\dot{y}_1,\ldots,\dot{y}_N) \qquad (4.1)\\
&\qquad\qquad (k=1,\ldots,N).
\end{aligned}
$$

We assume that the projections of resistance forces onto the x axis satisfy the conditions

$$R_k(0,\ldots,0,\dot{y}_1,\ldots,\dot{y}_N)\equiv 0 \quad (k=1,\ldots,N).$$

System (4.1) has the solution (unperturbed motion (1.3))

$$x_k\equiv 0,\quad y_k=y_{k0}(t)\quad (k=1,\ldots,N); \qquad (4.2)$$

at the same time $y_{k0}(t)$ satisfies the system of equations

$$
\begin{aligned}
\ddot{y}_{k0}+\frac{1}{m_k}R_{N+k}(0,\ldots,0,\dot{y}_{10},\ldots,\dot{y}_{N0})-p_ky_{k-1,0}&\\
+(p_k+\mu_{k+1}p_{k+1})y_{k0}-\mu_{k+1}p_{k+1}y_{k+1,0}&=0 \qquad (4.3)\\
(k=1,\ldots,N).&
\end{aligned}
$$

Here

$$p_k=\frac{c_k}{m_k}\quad (k=1,\ldots,N;\ p_{N+1}=0),$$

$$\mu_k=\frac{m_k}{m_{k-1}}\quad (k=2,\ldots,N;\ \mu_{N+1}=0).$$

Now we test whether conditions (1.7)-(1.10) ($n=2N$, $m=N$) are satisfied. Both (1.7) and (1.8) hold since

$$a_{\nu i}=\begin{cases} m_\nu\delta_{\nu i} & (\nu=1,\ldots,N;\ i=1,\ldots,2N),\\ m_{\nu-N}\delta_{\nu i} & (\nu=N+1,\ldots,2N;\ i=1,\ldots,2N).\end{cases}$$

Condition (1.9) requires that

$$\left(\frac{\partial R_k}{\partial \dot{y}_l}\right)_0=0,\quad \left(\frac{\partial R_{N+k}}{\partial \dot{x}_l}\right)_0=0\quad (k,\,l=1,\ldots,N), \qquad (4.4)$$

where the zero subscripts mean that after differentiation the values in (4.2) are substituted for the arguments. Conditions (4.4) hold, in particular, if R_k are independent of $\dot{y}_l$, and R_{N+k} are independent of $\dot{x}_l$ ($k,\ l=1,\ldots,N$). We assume that conditions (4.4) are satisfied.

For condition (1.10) to hold, it is necessary that

$$\left(\frac{\partial^2 \Pi}{\partial x_j \, \partial y_k}\right)_0 = 0 \quad (j, k = 1, \ldots, N)$$

be satisfied, since the zero subscript indicates, among other things, that after differentiation we set $x_1 = \ldots = x_N = 0$. Hence, the variational equations (1.11) and (1.12) hold for the perturbed motion ($x_k = 0 + \xi_k$, $y_k = y_{k0}(t) + \eta_k$; $k = 1, \ldots, N$)

$$\frac{d^2\xi_k}{dt^2} + \frac{1}{m_k} \sum_{i=1}^{N} \left[\left(\frac{\partial R_k}{\partial \dot{x}_i}\right)_0 \frac{d\xi_i}{dt} + \left(\frac{\partial^2 \Pi}{\partial x_i \, \partial x_k}\right)_0 \xi_i \right] = 0,$$

$$\frac{d^2\eta_k}{dt^2} + \frac{1}{m_k} \sum_{i=1}^{N} \left[\left(\frac{\partial R_{N+k}}{\partial \dot{y}_i}\right)_0 \frac{d\eta_i}{dt} + \left(\frac{\partial^2 \Pi}{\partial y_i \, \partial y_k}\right)_0 \eta_i \right] = 0$$

$$(k = 1, \ldots, N)$$

or, in detailed form,

$$\begin{aligned} \frac{d^2\xi_k}{dt^2} &+ \frac{1}{m_k} \sum_{i=1}^{N} \left(\frac{\partial R_k}{\partial \dot{x}_i}\right)_0 \frac{d\xi_i}{dt} - p_k \\ &\times \left\{ 1 - \left[1 + \gamma_k + \frac{1}{l_k} \left(y_{k0}(t) - y_{k-1,0}(t)\right) \right]^{-1} \right\} \\ &\times (\xi_{k-1} - \xi_k) + \mu_{k+1} p_{k+1} \left\{ 1 - \left[1 + \gamma_{k+1} + \frac{1}{l_{k+1}} \left(y_{k+1,0}(t) \right.\right.\right. \\ &\left.\left.\left. - y_{k0}(t)\right) \right]^{-1} \right\} (\xi_k - \xi_{k+1}) = 0, \end{aligned} \tag{4.5}$$

$$\begin{aligned} \frac{d^2\eta_k}{dt^2} &+ \frac{1}{m_k} \sum_{i=1}^{N} \left(\frac{\partial R_{N+k}}{\partial \dot{y}_i}\right)_0 \frac{d\eta_i}{dt} - p_k \eta_{k-1} + (p_k + \mu_{k+1} p_{k+1}) \eta_k \\ &- \mu_{k+1} p_{k+1} \eta_{k+1} = 0 \end{aligned} \tag{4.6}$$

$$(k = 1, \ldots, N).$$

Note that these equations are considered despite the fact that the stability in the large of the lower equilibrium position is already established by the theorem of Subsection 1.3, provided condition (1.1) is satisfied. The reasons are threefold: (1) there may be cases when (1.1) is violated (for example, in the case of partial dissipation); (2) equations (4.5)-(4.6) may be used to evaluate the stability of unperturbed motion (4.2); and (3) resistance forces may vanish. This conservative case is the subject of the next subsection.

1.5. Conservative case. With no resistance forces, a completely elastic free oscillatory chain becomes a conservative system. Deviations $y_{k0}(t)$ of its masses from the lower equilibrium position in

the course of vertical oscillations (unperturbed motion) satisfy system (4.3) when $R_{N+k} \equiv 0$ $(k = 1, \ldots, N)$ (this system describes small-amplitude oscillations of Sturm system [62]). The equation of frequencies ω of this system proves to be the secular equation of a Jacobian matrix

$$\begin{vmatrix} \omega^2-(p_1+\mu_2p_2) & \mu_2p_2 & 0 & \ldots & 0 \\ \mu_2p_2 & \omega^2-(p_2+\mu_3p_3) & \mu_3p_3 & \ldots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \ldots & \omega^2-p_N \end{vmatrix}=0.$$

When $R_j \equiv 0$ $(j = 1, \ldots, 2N)$, the coefficients of equations (4.5) from the system of equations for the first approximation of perturbed motion will be either periodic if the frequencies $\omega_1, \ldots, \omega_N$ are commensurable, or quasiperiodic if they are not. In both cases the investigation of the stability of unperturbed motion is a fairly complicated problem.

The analysis is facilitated by the fact that for the conservative system all solutions of (4.6) are bounded. This follows from the positivity of the eigenvalues of the Jacobian matrix given above. The boundedness of the solutions can also be established in a straightforward manner: for $R_{N+1} = R_{N+2} = \ldots = R_{2N} \equiv 0$ we recast system (4.6) in the form

$$\frac{d\eta_k}{dt} = \dot{\eta}_k,$$

$$\frac{d\dot{\eta}_k}{dt} = p_k\eta_{k-1} - (p_k + \mu_{k+1}p_{k+1})\,\eta_k + \mu_{k+1}p_{k+1}\eta_{k+1}$$

$$(k = 1, \ldots, N).$$

We consider a positive-definite quadratic form of the variables $\eta_1, \ldots, \eta_N, \dot{\eta}_1, \ldots, \dot{\eta}_N$ with constant coefficients

$$U = \frac{1}{2}\sum_{k=1}^{N} \mu_1 \ldots \mu_k\,[p_k(\eta_k - \eta_{k-1})^2 + \dot{\eta}_k^2] \quad (\mu_1 = 1,\ \eta_0 = 0).$$

By virtue of the above equations, its derivative is zero, thus establishing the boundedness of the solutions.

1.6. Stability of vertical vibrations of a spring-loaded pendulum. A single-link completely elastic free oscillatory chain is a pendulum formed by mass m suspended by a spring of unstressed length l

and with force constant c (Fig. 3). System (4.3) is reduced to a single equation

$$m\ddot{y}_0 + cy_0 = 0,$$

so that the unperturbed motion is

$$x \equiv 0, \quad y = y_0(t) = Y \cos \omega t \quad \left(\omega = \sqrt{\frac{c}{m}}\right).$$

The variational equations (4.5) and (4.6) become

$$\ddot{\xi} + \omega^2 \left(1 - \frac{1}{1+\gamma+\frac{1}{l} Y \cos \omega t}\right) \xi = 0, \quad \ddot{\eta} + \omega^2 \eta = 0. \tag{6.1}$$

Note that condition (1.8) is not satisfied as a result of internal properties of the mechanical system and the choice of unperturbed

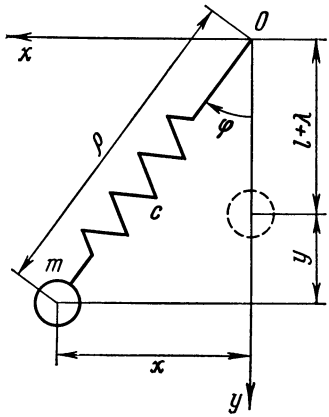

FIG. 3

motion; it is also determined by the choice of the Lagrangian coordinates. If we change to polar coordinates, we obtain

$$K = \frac{1}{2} m (\rho^2 \dot{\varphi}^2 + \dot{\rho}^2),$$

$$\Pi = \frac{1}{2} c (\rho - l)^2 - mg\rho \cos \varphi,$$

which transforms the Lagrange equations with no resistance forces to

$$\rho^2 \ddot{\varphi} + 2\rho\dot{\rho}\dot{\varphi} = -g\rho \sin \varphi,$$

$$\ddot{\rho} - \rho\dot{\varphi}^2 = -\frac{c}{m}(\rho - l) + g \cos \varphi.$$

In the notation used earlier, the vertical vibrations of mass m assumed to represent unperturbed motion are

$$\varphi = \varphi_0 \equiv 0, \quad \rho = \rho_0(t) = l + \lambda + Y \cos \omega t.$$

Formula (1.5) now gives for the coefficient $b_{11}(t)$ the expression

$$b_{11}(t) = 2m\rho_0\dot{\rho}_0 \not\equiv 0,$$

which states that condition (1.8) is violated. The variational equations for perturbed motion ($\varphi = 0 + \Phi$, $\rho = \rho_0(t) + \mathrm{P}$) become in polar coordinates

$$\frac{d^2\Phi}{d\tau^2} - 2\frac{\mu \sin\tau}{1+\gamma+\mu\cos\tau}\frac{d\Phi}{d\tau} + \frac{\gamma}{1+\gamma+\mu\cos\tau}\Phi = 0,$$

$$\frac{d^2\mathrm{P}}{d\tau^2} + \mathrm{P} = 0 \quad \left(\gamma = \frac{\lambda}{l},\ \mu = \frac{Y}{l},\ \tau = \omega t\right).$$

We observe that conservative systems may be described by variational equations containing a term with the first-order derivative.

By returning to Cartesian coordinates and introducing dimensionless time $\tau = \omega t$, we can present differential equations for perturbed motion in the form

$$\frac{d^2\xi}{d\tau^2} + \frac{\gamma+\mu\cos\tau}{1+\gamma+\mu\cos\tau}\xi + (2) - 0, \quad \frac{d^2\eta}{d\tau^2} + \eta + (2) = 0.$$

Here, as before, γ and μ stand for dimensionless parameters that are the ratios of static elongation and amplitude of vertical vibrations o unstressed spring length

$$\gamma = \frac{\lambda}{l}, \quad \mu \quad \frac{Y}{l},$$

and the (2)'s stand for second-order infinitesimals in γ and μ.

The stability (or instability) of the trivial solution of the first equation in (6.1) determines it for the whole system (6.1). In our case of a conservative system, however, the stability of the trivial solution of (6.1) does not determine the overall stability of unperturbed motion with respect to the variables x, y, $\dot{x}$, $\dot{y}$, because one of the critical situations arises. On the other hand, the instability of the trivial solution of (6 1) entails the instability (with the possible exception of boundary cases) of unperturbed motion ([40a], Sec. 70) with respect to x, y, $\dot{x}$, $\dot{y}$.

This is stipulated by the fact that the first variational equation has a periodic function for its coefficient, so that the lowest characteristic number, in the sense of Lyapunov, is negative when the trivial solution is unstable.

In order to analyze the instability of the trivial solution of the equation

$$\frac{d^2\xi}{d\tau^2} + \frac{\gamma+\mu\cos\tau}{1+\gamma+\mu\cos\tau}\xi = 0 \tag{6.2}$$

we start with the Zhukovskii criterion [400], which guarantees stability if the inequalities

$$\frac{1}{4}k^2 \leqslant p(t) \leqslant \frac{1}{4}(k+1)^2 \quad (k=0,\ 1,\ 2, \ldots)$$

are satisfied. Under the condition that $\mu < 1 + \gamma$, that is, when the amplitude of vertical vibrations is smaller than the length of a statically strained spring, we obtain

$$\inf p(\tau) = \frac{\gamma-\mu}{1+\gamma-\mu}, \quad \sup p(\tau) = \frac{\gamma+\mu}{1+\gamma+\mu}.$$

The Zhukovskii criterion requires that

$$\mu \leqslant \gamma, \quad \mu \leqslant \frac{1}{3} - \gamma \quad (\text{for } k=0),$$

or

$$\mu \leqslant -\frac{1}{3} + \gamma \quad (\text{for } k=1).$$

For $k > 1$ the criterion fails. The hatched area in Fig. 4 shows the resultant region of the stability of the trivial solution of (6.2) according to the Zhukovskii criterion. This diagram will be useful for comparison with the instability region.

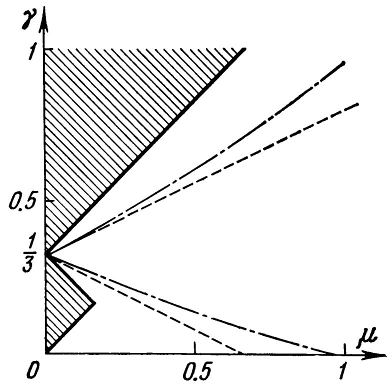

FIG. 4

In order to find the instability region by means of the method of small parameter, we choose μ as the small parameter and write

equation (6.2) in the form

$$\frac{d^2\xi}{d\tau^2}+[p_0(\gamma)+\mu p_1(\tau,\ \gamma)+\mu^2 p_2(\tau,\ \gamma)+\dots]\,\xi=0;$$

where

$$p_0(\gamma)=\frac{\gamma}{1+\gamma},$$

$$p_1(\tau,\ \gamma)=2p_1^{(1)}(\gamma)\cos\tau$$

$$\left(p_1^{(1)}(\gamma)=\frac{1}{2(1+\gamma)^2}\right).$$

In the scalar case, the instability regions in the $\mu\gamma$ plane may be contiguous on the axis $\mu = 0$ to those points γ_m that are roots of equation

$$2\sqrt{p_0(\gamma_m)}=m \quad \text{or} \quad \gamma_m=\frac{m^2}{4-m^2} \quad (m=1,\ 2,\ \dots) \qquad (6.3)$$

([80], V, 2.5). For $\gamma > 0$, a wide instability region (i.e. one with the angle between the tangents distinct from zero) is contiguous only to one point $\gamma_1 = \frac{1}{3}$ (this is the only such point on the half-axis $\gamma > 0$). The slope of the tangent in the above example is found by means of a formula derived from ([80], V, 2.24)

$$\chi^{\mp}=\mp\left[\frac{p_1^{(1)}(\gamma)}{dp_0/d\gamma}\right]_{\gamma=\gamma_1}=\mp\frac{1}{2}. \qquad (6.4)$$

This yields in the first approximation the instability region for vertical vibrations of a spring-loaded pendulum

$$\frac{1}{3}-\frac{1}{2}\mu+\dots<\gamma<\frac{1}{3}+\frac{1}{2}\mu+\dots.$$

The rays bounding this region are traced in Fig. 4 by dashed lines.

The general theory ([80], V, 2.3) shows that since

$$\left.\frac{dp_0}{d\gamma}\right|_{\gamma=\gamma_1}\neq 0,$$

the functions describing the boundaries must be analytic in the parameter μ; therefore, the terms omitted are of order not lower than μ^2. The next expansion coefficients can be found by using the fact that an antiperiodic solution (since m is odd) exists at the boundaries of this instability region. We give only the final result: in the second approximation the instability region is determined by the inequalities

$$\frac{1}{3}-\frac{1}{2}\mu+\frac{15}{128}\mu^2+\dots<\gamma<\frac{1}{3}+\frac{1}{2}\mu+\frac{15}{128}\mu^2+\dots. \qquad (6.5)$$

The boundaries of this region are given in Fig. 4 by dot-dash lines.

Experiments have demonstrated [394] that the perturbation of vertical vibrations of a spring-loaded pendulum with resistance forces occurs for $\lambda \approx \frac{1}{3} l$ $\left(\text{i.e. for } \gamma \approx \frac{1}{3}\right)$. In pendulums with dissipation, "conservative instability" regions result in the instability of vertical vibrations. Despite the asymptotic damping of vibrations, which proceeds slowly if dissipation is not too large, resonance in a chain may cause substantial changes in the vibrations and become essential in evaluating a system's performance.

§ 2. Partly Elastic Free Oscillatory Chains

The theory of stability of vertical oscillations is an important part of the general theory of oscillatory chains. The analysis of stability is carried out by means of the mathematical theory of parametric resonance. It should be emphasized that the problems we wish to discuss sometimes involve canonical (Hamiltonian) systems of linear differential equations with periodic coefficients of a special type, namely, where the parameter γ, which is equal to the inverse frequency of parametric excitation, appears nonlinearly. Expressions for the boundaries of the dynamic instability regions in such systems were obtained by Jakubovich and Pittel ([339]; [80], V, 2.3).

2.1. Statement of the problem. We consider a mechanical system comprising N point masses with mass m_k and rectangular Cartesian coordinates x_k, y_k, z_k $(k = 1, \ldots, N)$ with respect to an inertial reference frame. Of $3N - n$ holonomic constraints that do not depend on time explicitly, let N constraints be $z_1 = 0, \ldots, z_N = 0$ (this means that the motion takes place in a plane) and $2N - n$ constraints be

$$f_\alpha(x_1, y_1, \ldots, x_N, y_N) = 0 \quad (\alpha = 1, 2, \ldots, 2N - n).$$

We denote the Lagrangian coordinates of the system by $q_1, \ldots, q_n$ and write the generalized force corresponding to a coordinate q_ν by

$$Q_\nu(q_1, \ldots, q_n) - R_\nu(\dot{q}_1, \ldots, \dot{q}_n) \quad (\nu = 1, \ldots, n), \tag{1.1}$$

where Q_ν and R_ν are continuous and differentiable functions of their respective arguments in the domains of their definition.

The kinetic energy of the system is

$$K = \frac{1}{2} \sum_{i,\, j=1}^{n} a_{ij} \dot{q}_i \dot{q}_j;$$

where

$$a_{ji} = a_{ij}(q_1, \ldots, q_n) = \sum_{k=1}^{n} m_k \left(\frac{\partial x_k}{\partial q_i} \frac{\partial x_k}{\partial q_j} + \frac{\partial y_k}{\partial q_i} \frac{\partial y_k}{\partial q_j} \right) \quad (i, j = 1, \ldots, n).$$

The equations of motion of the second kind in the Lagrange form become

$$\sum_{i=1}^{n} a_{\nu i}\ddot{q}_i + \sum_{i,\,j=1}^{n}\sum_{k=1}^{N} m_k\left(\frac{\partial^2 x_k}{\partial q_i\,\partial q_j}\frac{\partial x_k}{\partial q_\nu} - \frac{1}{2}\frac{\partial^2 x_k}{\partial q_i\,\partial q_\nu}\frac{\partial x_k}{\partial q_j} + \frac{1}{2}\frac{\partial^2 x_k}{\partial q_\nu\partial q_j}\frac{\partial x_k}{\partial q_i}\right.$$
$$\left. + \frac{\partial^2 y_k}{\partial q_i\,\partial q_j}\frac{\partial y_k}{\partial q_\nu} - \frac{1}{2}\frac{\partial^2 y_k}{\partial q_i\,\partial q_\nu}\frac{\partial y_k}{\partial q_j} + \frac{1}{2}\frac{\partial^2 y_k}{\partial q_\nu\partial q_j}\frac{\partial y_k}{\partial q_i}\right)\dot{q}_i\dot{q}_j = Q_\nu - R_\nu \quad (1.2)$$
$$(\nu = 1,\ \ldots,\ n).$$

In order to analyze the stability of unperturbed motion

$$q_\nu = q_{\nu 0}(t), \quad \dot{q}_\nu = \dot{q}_{\nu 0}(t) \quad (\nu = 1,\ \ldots,\ n) \quad (1.3)$$

with respect to the variables

$$q_1,\ \ldots,\ q_r,\ \dot{q}_1,\ \ldots,\ \dot{q}_r \quad (r \leqslant n)$$

we denote the coordinates and velocities of perturbed motion by

$$q_\nu = q_{\nu 0}(t) + \varkappa_\nu, \quad \dot{q}_\nu = \dot{q}_{\nu 0}(t) + \dot{\varkappa}_\nu \quad (\nu = 1,\ \ldots,\ n).$$

The variational equations of perturbed motion can be given in the first approximation as

$$\sum_{i=1}^{n}\left[(a_{\nu i})_0\frac{d^2\varkappa_i}{dt^2} + b_{\nu i}(t)\frac{d\varkappa_i}{dt} + c_{\nu i}(t)\,\varkappa_i\right] = 0 \quad (\nu = 1,\ \ldots,\ n). \quad (1.4)$$

Here

$$b_{\nu i}(t) = 2\sum_{k=1}^{N} m_k\sum_{j=1}^{n}\left(\frac{\partial^2 x_k}{\partial q_i\,\partial q_j}\frac{\partial x_k}{\partial q_\nu} + \ldots\right)\dot{q}_{j0}(t) + \left(\frac{\partial R_\nu}{\partial \dot{q}_i}\right)_0,$$

$$c_{\nu i}(t) = \sum_{k=1}^{N} m_k\sum_{j=1}^{n}\left\{\left(\frac{\partial^2 x_k}{\partial q_\nu\partial q_i}\frac{\partial x_k}{\partial q_j} + \frac{\partial^2 x_k}{\partial q_i\,\partial q_j}\frac{\partial x_k}{\partial q_\nu} + \ldots\right)\ddot{q}_{j0}(t)\right.$$
$$+ \sum_{l=1}^{n}\left[\frac{1}{2}\frac{\partial^3 x_k}{\partial q_\nu\,\partial q_i\,\partial q_j}\frac{\partial x_k}{\partial q_l} + \frac{1}{2}\frac{\partial^2 x_k}{\partial q_\nu\partial q_j}\frac{\partial^2 x_k}{\partial q_i\,\partial q_l}\right.$$
$$+ \frac{\partial^2 x_k}{\partial q_\nu\,\partial q_i}\frac{\partial^2 x_k}{\partial q_j\,\partial q_l} + \frac{\partial^3 x_k}{\partial q_i\,\partial q_j\,\partial q_l}\frac{\partial x_k}{\partial q_\nu} - \frac{1}{2}\frac{\partial^3 x_k}{\partial q_\nu\,\partial q_i\,\partial q_l}\frac{\partial x_k}{\partial q_j}$$
$$\left.\left. - \frac{1}{2}\frac{\partial^2 x_k}{\partial q_\nu\,\partial q_l}\frac{\partial^2 x_k}{\partial q_i\,\partial q_j} + \ldots\right]\dot{q}_{j0}(t)\,\dot{q}_{l0}(t)\right\} - \left(\frac{\partial Q_\nu}{\partial q_i}\right)_0$$
$$(\nu = 1,\ \ldots,\ n).$$

The zero subscript on $a_{\nu i}$ and on the partial derivatives denotes evaluation at $q_{10}(t),\ \ldots,\ q_{n0}(t);\ \dot{q}_{10}(t),\ \ldots,\ \dot{q}_{n0}(t)$. In the expres-

sions for $b_{\nu i}(t)$ and $c_{\nu i}(t)$, terms obtained by substituting y for x are ignored.

In Subsection 1.1 we introduced the definition that a mechanical system is an "oscillatory chain" with respect to unperturbed motion (1.3) if it is possible to choose the Lagrangian coordinates for which the coefficients $(a_{\nu i})_0$, $b_{\nu i}(t)$, and $c_{\nu i}(t)$ are such that conditions (1, 1.7)-(1, 1.10) are satisfied for some $m < n$ (m is a natural number) and $t \geqslant t_0$.

Conditions (1, 1.7)-(1, 1.10) mean that the variational equations (1.4) fall into two groups (1, 1.11) and (1, 1.12) of m and $n - m$ equations.

The kinetic energy K is positive and therefore equations (1, 1.11) and (1, 1.12) are solvable with respect to the derivatives of higher order.

2.2. Kinetic and potential energies. We consider a completely elastic free oscillatory chain consisting of N masses $m_1, \ldots, m_N$ connected in series by N weightless springs with force constants $c_1, \ldots, c_N$ and of unstressed lengths $l_1, \ldots, l_N$ (see Fig. 2). The first spring is fixed at point O, and each subsequent spring is attached to a weightless hinge whose axis determines the motion in a plane. Let a group of h springs be replaced by rigid weightless rods; let these be the rods connecting the masses $m_f, m_{f+1}, \ldots, m_{f+h}$. In addition to N trivial constraints $z_1 = 0, \ldots, z_N = 0$, h new constraints are imposed on the system:

$$(x_{f+\eta} - x_{f+\eta-1})^2 + (y'_{f+\eta} - y'_{f+\eta-1})^2 = l^2_{f+\eta} \quad (\eta = 1, \ldots, h),$$

where x_ν, y'_ν are the Cartesian coordinates of the νth mass m_ν ($\nu = 1, \ldots, N$; $x_0 = y'_0 = 0$). In this situation the abscissas (or ordinates) of the masses $m_{f+1}, \ldots, m_{f+h}$ are no longer Lagrangian coordinates. Assume for $2N - h$ coordinates that

$$x_1, \ldots, x_f; \quad \varphi_{f+1}, \ldots, \varphi_{f+h}; \quad x_{f+h+1}, \ldots, x_N;$$

$$y_1 = y'_1 - L_1,$$

$$\ldots\ldots\ldots\ldots\ldots$$

$$y_f = y'_f - (L_1 + \ldots + L_f),$$

$$y_{f+h+1} = y'_{f+h+1} - (L_1 + \ldots + L_{f+h+1}),$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$y_N = y'_N - (L_1 + \ldots + L_N).$$

Here L_k is the length of the kth spring in the strained state ($k = 1, \ldots, f, f + h + 1, \ldots, N$), or that of a rigid rod ($k = f + 1, \ldots, f + h$), i.e. $L_k = l_k + \lambda_k$, where λ_k is the static

elongation of the kth spring

$$\lambda_k = (m_k + m_{k+1} + \ldots + m_N)\, g/c_k$$

$$(k = 1, \ldots, f, f + h + 1, \ldots, N;\ \lambda_{f+1} = \ldots = \lambda_{f+h} = 0).$$

The coordinates $y_1, \ldots, y_f, y_{f+h+1}, \ldots, y_N$ are thus equal to the displacements of the corresponding masses along the vertical axis from the lower equilibrium position. The coordinates $\varphi_{f+1}, \ldots, \varphi_{f+h}$ are polar angles measured clockwise from the y axis to the rods $l_{f+1}, \ldots, l_{f+h}$.

The kinetic energy K of the system is

$$K = \frac{1}{2} \sum_{k=1}^{N}{}' m_k (\dot{x}_k^2 + \dot{y}_k^2) + \frac{1}{2} \sum_{\eta=1}^{h} m_{f+\eta} \Big\{ (\dot{x}_f^2 + \dot{y}_f^2)$$

$$+ \sum_{\alpha=1}^{\eta} l_{f+\alpha}\dot{\varphi}_{f+\alpha} [l_{f+\alpha}\dot{\varphi}_{f+\alpha} + 2 (\dot{x}_f \cos \varphi_{f+\alpha} - \dot{y}_f \sin \varphi_{f+\alpha})]$$

$$+ 2 \sum_{\substack{\alpha, \beta=1 \\ \alpha<\beta}}^{n} l_{f+\alpha} l_{f+\beta} \dot{\varphi}_{f+\alpha} \dot{\varphi}_{f+\beta} \cos (\varphi_{f+\alpha} - \varphi_{f+\beta}) \Big\}. \tag{2.1}$$

We express the potential energy Π of the linear elastic forces of the springs and of the forces of gravity as

$$\Pi = -g \sum_{\alpha=1}^{f} (m_{f+1} + \ldots + m_{f+h} + m_\alpha)\, y_\alpha$$

$$- g \sum_{\beta=f+h+1}^{N} m_\beta y_\beta - g \sum_{\eta=1}^{h} m_{f+\eta} l_{f+\eta} \cos \varphi_{f+\eta}$$

$$+ \frac{1}{2} \sum_{k=1}^{N}{}' c_k [(x_k - x_{k-1})^2 + (l_k + \lambda_k + y_k - y_{k-1})^2]$$

$$- \sum_{k=1}^{N}{}' c_k l_k \sqrt{(x_k - x_{k-1})^2 + (l_k + \lambda_k + y_k - y_{k-1})^2} \tag{2.2}$$

assuming that $x_{f+h} = l_{f+h} \sin \varphi_{f+h}$, $y_{f+h} = l_{f+h} (\cos \varphi_{f+h} - 1)$ and always taking the arithmetic value of the square root. The sum $\sum'$ is primed to indicate that the addends corresponding to $k = f + 1, \ldots, f + h$ must be dropped. It is readily demonstrated, by analogy with Subsection 1.2, that there exist 2^N equilibrium positions in a partly elastic free oscillatory chain. The theorem of Subsection 1.3 also holds: the lower equilibrium position

(corresponding to zero values of the Lagrangian coordinates) is asymptotically stable in the large if the work of resistance forces is negative for any virtual displacement.

Now we denote the Lagrangian coordinates in the order in which they were introduced by $q_1, \ldots, q_{2N-h}$. This yields for Q_ν in equalities (1.1)

$$Q_\nu = \frac{\partial \Pi}{\partial q_\nu} \quad (\nu = 1, \ldots, n).$$

We assume for the resistance forces that

$$\begin{aligned}
&R_k = R_k(\dot{q}_1, \ldots, \dot{q}_N) && (k = 1, \ldots, N),\\
&R_{N+j} = R_{N+j}(\dot{q}_{N+1}, \ldots, \dot{q}_{2N-h}) && (j = 1, \ldots, N-h),\\
&R_\nu(0, \ldots, 0) = 0 && (\nu = 1, \ldots, 2N-h).
\end{aligned}$$

One possible solution of equations (1.2) (vertical oscillations of a partly elastic free oscillatory chain) is

$$\begin{aligned}
q_k &= q_{k0}(t) \equiv 0 \quad && (k = 1, \ldots, N),\\
q_{N+j} &= q_{N+j,\,0}(t) && (j = 1, \ldots, N-h),
\end{aligned}$$

where $q_{N+j,\,0}(t)$ is determined by the last equations of (1.2). We assume this solution to be the unperturbed motion (1.3). It is readily shown that conditions (1, 1.7)-(1, 1.10) are satisfied, and therefore the variational equations fall into two groups (1, 1.11) and (1, 1.12) of N and $N - h$ equations. The instability of unperturbed motion is determined by the instability of the trivial solution of the first group (1, 1.11) because the characteristic numbers of the second group (1, 1.12) are nonnegative (see Subsection 1.6).

2.3. Example. We consider a two-link oscillatory chain where the first link is absolutely rigid and the second is elastic with force constant c. The notation is clear from Fig. 5. The static elongation of the spring is $\lambda_2 = m_2 g/c$. Formulas (2.1) and (2.2) become

$$\begin{aligned}
K &= \frac{1}{2} m_2 (\dot{x}_2^2 + \dot{y}_2^2) + \frac{1}{2} m_1 l_1 \dot{\varphi}_1^2,\\
\Pi &= -m_1 g l_1 \cos\varphi_1 - m_2 g y_2\\
&\quad + \frac{1}{2} c\,[(x_2 - l_1 \sin\varphi_1)^2 + (l_2 + \lambda_2 + y_2 + l_1 - l_1\cos\varphi_1)^2]\\
&\quad - c l_2\,[(x_2 - l_1 \sin\varphi_1)^2 + (l_2 + \lambda_2 + y_2 + l_1 - l_1 \cos\varphi_1)^2]^{1/2}.
\end{aligned}$$

With the natural assumption $l_2 + \lambda_2 + y_2 > 0$, the Lagrange equations (1.2) have a solution (taking the vertical oscillations of mass m_2 as the unperturbed motion)

$$\varphi_1 = 0, \quad x_2 = 0, \quad y_2 = Y \cos\Omega t \quad \left(\Omega = \sqrt{\frac{c}{m_2}}\right).$$

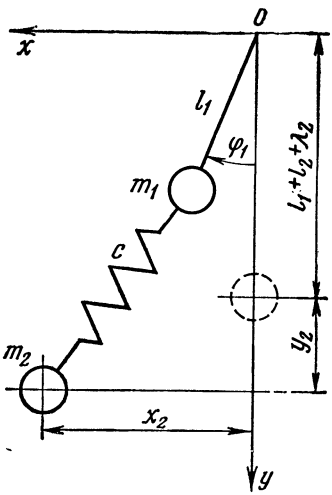

FIG. 5

The variational equations (1, 1.11) (for $\varphi_1 = 0 + \Phi$, $x_2 = 0 + \varkappa_2$) can be presented in dimensionless form

$$\frac{d^2\Phi}{d\tau^2}+\left[(1+\alpha)\beta\gamma+\alpha\beta\mu\cos\tau+\alpha\left(1-\frac{1}{1+\gamma+\mu\cos\tau}\right)\right]\Phi$$

$$-\sqrt{\alpha}\left(1-\frac{1}{1+\gamma+\mu\cos\tau}\right)\xi=0, \quad (3.1)$$

$$\frac{d^2\xi}{d\tau^2}-\sqrt{\alpha}\left(1-\frac{1}{1+\gamma+\mu\cos\tau}\right)\Phi+\left(1-\frac{1}{1+\gamma+\mu\cos\tau}\right)\xi=0$$

$$\left(\tau=\Omega t,\quad \xi=\sqrt{\alpha}\frac{\varkappa_2}{l_1},\quad \alpha=\frac{m_2}{m_1},\quad \beta=\frac{l_2}{l_1},\quad \gamma=\frac{\lambda_2}{l_2},\quad \mu=\frac{Y}{l_2}\right).$$

Limiting the analysis to the case $m_1 = m_2$, $l_1 = l_2$ ($\alpha = \beta = 1$), we rewrite system (3.1) in matrix form

$$\frac{d^2\mathbf{y}}{d\tau^2}+[\mathbf{P}_0(\gamma)+\mu\mathbf{P}_1(\tau,\ \gamma)+\mu^2\mathbf{P}_2(\tau,\ \gamma)+\ldots]\,\mathbf{y}=0,$$

where

$$\mathbf{y}=\begin{pmatrix}\Phi\\ \xi\end{pmatrix},\qquad \mathbf{P}_0(\gamma)=\frac{\gamma}{1+\gamma}\begin{Vmatrix}1+2(1+\gamma) & -1\\ -1 & 1\end{Vmatrix},$$

$$\mathbf{P}_1(\tau,\ \gamma)=2\mathbf{P}_1^{(1)}(\gamma)\cos\tau,\qquad \mathbf{P}_1^{(1)}(\gamma)=\frac{1}{2(1+\gamma)^2}\begin{Vmatrix}1+(1+\gamma)^2 & -1\\ -1 & 1\end{Vmatrix}.$$

The matrix

$$\frac{d\mathbf{P}_0}{d\gamma}=\frac{1}{(1+\gamma)^2}\begin{Vmatrix}1+2\,(1+\gamma)^2 & -1\\ -1 & 1\end{Vmatrix}$$

is positive-definite, and the results of Jakubovich and Pittel are therefore applicable ([339]; [80], V, 2.3). Let us calculate $\omega_1(\gamma)$ and $\omega_2(\gamma)$, that is, the natural frequencies of the unperturbed system

$$\omega_1=\sqrt{\gamma\left[1+\frac{1}{1+\gamma}-\sqrt{1+\frac{1}{(1+\gamma)^2}}\right]},$$

$$\omega_2=\sqrt{\gamma\left[1+\frac{1}{1+\gamma}+\sqrt{1+\frac{1}{(1+\gamma)^2}}\right]}. \qquad (3.2)$$

Wide instability regions (those where the angle between the tangents is distinct from zero) may be contiguous only to those points of the half-axis $\mu=0$ $(\gamma>0)$ for which

$$\omega_j(\gamma_0)+\omega_h(\gamma_0)=1 \quad (j,\ h=1,\ 2). \qquad (3.3)$$

Since in the example under discussion the functions

$$2\omega_1(\gamma),\ 2\omega_2(\gamma) \quad \text{and} \quad \omega_1(\gamma)+\omega_2(\gamma)$$

increase with γ, wide instability regions can exist for a single value of γ in each of the following three cases: (1) $2\omega_1(\gamma)=1$ and (2) $2\omega_2(\gamma)=1$ (principal resonances); and (3) $\omega_1(\gamma)+\omega_2(\gamma)=1$ (combination resonance). Since $(\mathbf{P}_1)_{\text{av}}=0$, we can use formula (V, 2.24) of [80] to calculate $\chi^{\mp}$ (the slopes of the tangents to the boundaries of the instability regions)

$$\chi^{\mp}=\mp\left.\frac{2\,(\mathbf{P}_1^{(1)}\mathbf{a}_j,\ \mathbf{a}_h)}{\dfrac{d}{d\gamma}(\omega_j+\omega_h)}\right|_{\gamma=\gamma_0}. \qquad (3.4)$$

Here γ_0 is the root of the corresponding equation

$$\omega_j(\gamma)+\omega_h(\gamma)=1 \quad (j,\ h=1,\ 2),$$

and $\mathbf{a}_j$ and $\mathbf{a}_h$ are the corresponding eigenvectors of the matrix $\mathbf{P}_0$, that is,

$$\mathbf{P}_0(\gamma_0)\,\mathbf{a}_j=\omega_j^2(\gamma_0)\,\mathbf{a}_j,\quad \mathbf{P}_0(\gamma_0)\,\mathbf{a}_h=\omega_h^2(\gamma_0)\,\mathbf{a}_h,$$

and

$$[\omega_j(\gamma_0)+\omega_h(\gamma_0)]\,(\mathbf{a}_j,\ \mathbf{a}_h)=\delta_{jh} \quad (j,\ h=1,\ 2).$$

The results of the calculations are given below. The regions of the first (*1*) and second (*2*) principal resonances (Fig. 6) are determined to the accuracy of $O(\mu^2)$ from the inequalities

$$0.076 - 0.364\mu + \ldots < \gamma < 0.076 + 0.364\mu + \ldots,$$
$$0.549 - 0.385\mu + \ldots < \gamma < 0.549 + 0.385\mu + \ldots,$$

and the region of the combination resonance (*3*) is derived from

$$0.157 - 0.088\mu + \ldots < \gamma < 0.157 + 0.088\mu + \ldots .$$

2.4. Pendulum subject to elastic free suspension. Let a simple pendulum of mass m_2 and rod length l_2 be suspended from a hinge of mass m_1 (Fig. 7). A spring of unstressed length l_1 and with force

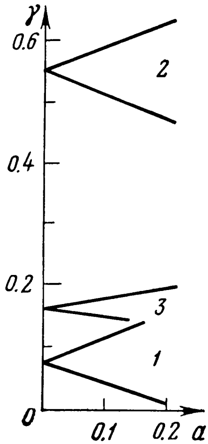

FIG. 6

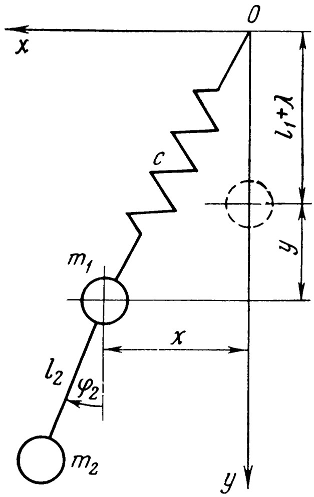

FIG. 7

constant c connects the hinge to a fixed point O; the static elongation of the spring is $\lambda = (m_1 + m_2)\, g/c$. Formulas (2.1) and (2.2) yield

$$K = \frac{1}{2}(m_1 + m_2)(\dot{x}^2 + \dot{y}^2) + \frac{1}{2} m_2 l_2 \dot{\varphi}^2 + m_2 l_2 \dot{\varphi}(\dot{x}\cos\varphi - \dot{y}\sin\varphi),$$

$$\Pi = -(m_1 + m_2)\, gy - m_2 g l_2 \cos\varphi$$
$$+ \frac{1}{2} c\,[x^2 + (l_1 + \lambda + y)^2] - c l_1 \sqrt{x^2 + (l_1 + \lambda + y)^2}.$$

Equations (1.2) take the form

$$(m_1 + m_2)\,\ddot{x} + m_2 l_2 \ddot{\varphi}\cos\varphi - m_2 l_2 \dot{\varphi}^2 \sin\varphi$$
$$= -cx + c l_1 x\,[x^2 + (l_1 + \lambda + y)^2]^{-1/2},$$

$$(m_1+m_2)\ddot{y}-m_2l_2\ddot{\varphi}\sin\varphi-m_2l_2\dot{\varphi}^2\cos\varphi$$
$$=-c(l_1+y)+cl_1(l_1+\lambda+y)[x^2+(l_1+\lambda+y)^2]^{-1/2},$$
$$l_2\ddot{\varphi}+\ddot{x}\cos\varphi-\ddot{y}\sin\varphi=-g\sin\varphi. \tag{4.1}$$

Let us introduce dimensionless coordinates $\xi = x/l_1$ and $\eta = y/l_1$ and dimensionless time $\tau = \sqrt{c/(m_1+m_2)}\, t$; this transforms the system of equations of motion to

$$\xi''+\alpha\beta\varphi''\cos\varphi=\alpha\beta\varphi'^2\sin\varphi-\xi+\xi[\xi^2+(1+\gamma+\eta)^2]^{-1/2},$$
$$\eta''-\alpha\beta\varphi''\sin\varphi$$
$$=\alpha\beta\varphi'^2\cos\varphi-(1-\eta)+(1+\gamma+\eta)[\xi^2+(1+\gamma+\eta)^2]^{-1/2},$$
$$\xi''\cos\varphi-\eta''\sin\varphi+\beta\varphi''=-\gamma\sin\varphi,$$

where

$$\alpha=\frac{m_2}{m_1+m_2}<1,\quad \beta=\frac{l_2}{l_1},\quad \gamma=\frac{\lambda}{l_1},$$

and the derivatives with respect to τ are denoted by primes. Solving the last system for higher derivatives, we obtain

$$\xi''=-\frac{1-\alpha\sin^2\varphi}{1-\alpha}\mathcal{R}\xi+\alpha\beta\varphi'^2\sin\varphi+\frac{\alpha}{1-\alpha}(1+\gamma+\eta)\mathcal{R}\sin\varphi\cos\varphi,$$
$$\eta''=\gamma-\frac{1-\alpha\cos^2\varphi}{1-\alpha}\mathcal{R}(1+\gamma+\eta)+\alpha\beta\varphi'^2\cos\varphi$$
$$+\frac{\alpha}{1-\alpha}\mathcal{R}\xi\sin\varphi\cos\varphi,$$
$$\varphi''=\frac{1}{\beta(1-\alpha)}\mathcal{R}\xi\cos\varphi-\frac{1+\gamma+\eta}{\beta(1-\alpha)}\mathcal{R}\sin\varphi, \tag{4.2}$$

where

$$\mathcal{R}\equiv 1-[\xi^2+(1+\gamma+\eta)^2]^{-1/2}.$$

The equations of motion (4.2) have a solution

$$\xi=\varphi\equiv 0,\quad \eta=\mu\cos\tau\quad (\mu>0) \tag{4.3}$$

These vertical oscillations of masses m_1 and m_2 are taken as unperturbed motion. The first group of variational equations (1, 1.11) ($\xi = 0 + u$, $\varphi = 0 + \psi$, $\eta = \mu\cos\tau + w$) may be presented in the form

$$\mathbf{y}''+\mathbf{Q}(\tau,\ \gamma,\ \mu;\ \alpha,\ \beta)\,\mathbf{y}=\mathbf{0}, \tag{4.4}$$

where

$$y=\binom{\psi}{u}, \quad Q=\frac{\gamma+\mu\cos\tau}{1-\alpha}\begin{Vmatrix} \frac{1}{\beta} & -\frac{1}{\beta(1+\gamma+\mu\cos\tau)} \\ -\alpha & \frac{1}{1+\gamma+\mu\cos\tau}\end{Vmatrix},$$

and the second group (1, 1.12) consists of the single equation

$$w''+w=0.$$

The instability of unperturbed motion (4.2) is determined by the instability of the trivial solution of system (4.4), which, if we assume $\mu<1+\gamma$, can be presented in the form

$$\mathbf{y}''+[\mathbf{Q}_0(\gamma;\alpha,\beta)+\mu\mathbf{Q}_1(\tau,\gamma;\alpha,\beta)+O(\mu^2)]\,\mathbf{y}=\mathbf{0}, \tag{4.5}$$

where

$$\mathbf{Q}_0=\frac{\gamma}{1-\alpha}\begin{Vmatrix} \frac{1}{\beta} & -\frac{1}{\beta(1+\gamma)} \\ -\alpha & \frac{1}{1+\gamma}\end{Vmatrix},$$

$$\mathbf{Q}_1=2\mathbf{Q}_1^{(1)}\cos\tau, \quad \mathbf{Q}_1^{(1)}=\frac{1}{1-\alpha}\begin{Vmatrix} \frac{1}{\beta} & -\frac{1}{\beta(1+\gamma)^2} \\ -\alpha & \frac{1}{(1+\gamma)^2}\end{Vmatrix}.$$

The eigenvalues ω_1^2 and ω_2^2 of the matrix $\mathbf{Q}_0$ are positive and equal to

$$\omega_{1,2}^2=\frac{\gamma}{2(1-\alpha)}\left[\frac{1}{\beta}+\frac{1}{1+\gamma}\mp\sqrt{\frac{1}{\beta^2}+\frac{1}{(1+\gamma)^2}+\frac{2(2\alpha-1)}{\beta(1+\gamma)}}\right].$$

Formulas V, 2.3 of [80] cannot be directly applied, however, to system (4.5), because the matrix $\mathbf{Q}$ in (4.4) (and consequently the matrices $\mathbf{Q}_0$ and $\mathbf{Q}_1^{(1)}$ in (4.5)) is not symmetric; systems (4.4) and (4.5) are "entangled" by the linear substitution. System (4.5) must be preliminarily "disentangled" by the linear transformation

$$\mathbf{y}=[\mathbf{S}_0(\gamma;\alpha,\beta)+\mu\mathbf{S}_1(\gamma;\alpha,\beta)+O(\mu^2)]\,\mathbf{v}$$
$$(\det\mathbf{S}_0\neq 0, \quad \det\mathbf{S}_1\neq 0).$$

The matrices $\mathbf{P}_0$ and $\mathbf{P}_1^{(1)}$ of the transformed system

$$\mathbf{v}''+[\mathbf{P}_0(\gamma;\alpha,\beta)+\mu\cdot 2\mathbf{P}_1^{(1)}(\gamma;\alpha,\beta)\cos\tau+O(\mu^2)]\,\mathbf{v}=\mathbf{0} \tag{4.6}$$

are real and symmetric, for example,

$$\mathbf{P}_0=\mathbf{S}_0^{-1}\mathbf{Q}_0\mathbf{S}_0=\begin{Vmatrix}\omega_1^2 & 0\\ 0 & \omega_2^2\end{Vmatrix}.$$

Formulas V, 2.3 of [80] are applicable to the transformed system to determine the boundaries of the dynamic instability regions in the first approximation with respect to μ. In the special case discussed in Subsection 4.3 ($m_1 = m_2$, $l_1 = l_2$, i.e. $\alpha = \frac{1}{2}$, $\beta = 1$) we arrive at the same expressions (3.2) for $\omega_1(\gamma)$ and $\omega_2(\gamma)$, and hence the same critical values of the parameter γ, which are contiguous to the two regions of fundamental and one region of combination resonance. The calculation of the slopes $\chi^{\mp}$ of the tangents to these regions will not be discussed here.

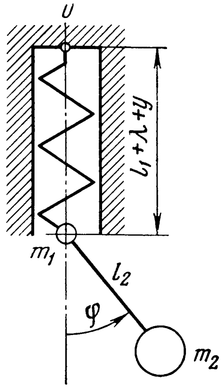

FIG. 8

2.5. Pendulum subject to elastic guided suspension. Let the motion of mass m_1 be constrained by vertical guides (Fig. 8); the notation remains the same as in the preceding subsection. This oscillatory chain is not free (one degree of freedom corresponding to x is lost) and partly elastic. The kinetic and potential energies are now given by

$$K = \frac{1}{2}(m_1 + m_2)\dot{y}^2 + \frac{1}{2} m_2 l_2^2 \dot{\varphi}^2 - m_2 l_2 \dot{y}\dot{\varphi} \sin\varphi,$$

$$\Pi = \frac{1}{2} c y^2 + m_2 g l_2 (1 - \cos\varphi) \tag{5.1}$$

and the Lagrange equations of the second kind are

$$(m_1 + m_2)\ddot{y} - m_2 l_2 \ddot{\varphi} \sin\varphi - m_2 l_2 \dot{\varphi}^2 \cos\varphi = -cy,$$

$$-\ddot{y}\sin\varphi + l_2\ddot{\varphi} = -g\sin\varphi.$$

Solving the last equations with respect to higher derivatives, we obtain

$$\eta'' + \eta = (1 - \alpha\sin^2\varphi)^{-1}(\alpha\varphi'^2\cos\varphi - \alpha\gamma\sin^2\varphi - \alpha\eta\sin^2\varphi),$$

$$\varphi'' + \gamma\varphi = (1 - \alpha\sin^2\varphi)^{-1}(-\eta\sin\varphi + \alpha\varphi'^2\sin\varphi\cos\varphi - \alpha\gamma\sin^3\varphi) + \gamma(\varphi - \sin\varphi), \tag{5.2}$$

where the following dimensionless variables and parameters are used

$$\eta = \frac{y}{l_2}, \quad \tau = \omega t \quad \left(\omega = \sqrt{\frac{c}{m_1 + m_2}}\right),$$

$$\alpha = \frac{m_2}{m_1 + m_2} < 1, \quad \gamma = \frac{\lambda}{l_2}. \tag{5.3}$$

Let us analyze the stability of unperturbed motion, in this case, of vertical oscillations

$$\varphi \equiv 0, \quad \eta = \mu \cos \tau \quad (\mu > 0). \tag{5.4}$$

Assuming that for perturbed motion $\varphi = 0 + \psi$, $\eta = \mu \cos \tau + v$, we obtain for the first group of variational equations (1, 1.11) the single equation

$$\psi'' + (\gamma + \mu \cos \tau)\, \psi = 0, \tag{5.5}$$

and for the second group (1, 1.12) the single equation

$$v'' + v = 0.$$

The instability of unperturbed motion (5.4) is determined by the instability of the trivial solution of equation (5.5) (Mathieu's equation). The literature devoted to Mathieu's equation is quite extensive; for our purposes it is sufficient to apply equation (3.3) and formula (3.4) when the matrices $\mathbf{P}_0$ and $\mathbf{P}_1^{(1)}$ degenerate to the scalars γ and $\frac{1}{2}$, respectively. Equation (3.3) yields the critical values of the parameter

$$2\omega(\gamma_n) \equiv 2\sqrt{\gamma_n} = n, \quad \gamma_n = \frac{1}{4} n^2 \quad (n = 1, 2, \ldots),$$

and (3.4) gives

$$\chi^{\mp} = \mp \left. \frac{P_1^{(1)}}{\dfrac{d\omega^2}{d\gamma}} \right|_{\gamma = \gamma_1} = \mp \frac{1}{2} \tag{5.6}$$

for the slopes of the tangents to the wide instability region in the $\mu\gamma$ plane. This region (i.e. the region with the angle between the tangents distinct from zero) is contiguous to a single point $\gamma = \frac{1}{4}$ and in the first approximation is determined by the inequalities

$$\frac{1}{4} - \frac{1}{2}\mu + \ldots < \gamma < \frac{1}{4} + \frac{1}{2}\mu + \ldots . \tag{5.7}$$

CHAPTER III

APPLICATION OF THE METHODS OF SMALL PARAMETER TO OSCILLATIONS IN LYAPUNOV SYSTEMS

This chapter treats several problems in mechanics and physics. The methods of small parameter provide suitable mathematical techniques for determining periodic motions and describing transient processes if the periodic motion corresponds to the limit cycle (in phase space of dimension $2k > 2$). Computational aspects of the Poincaré method [188a] are presented in the first two sections; as for the method of averaging, only the simplest applications in the sense of Van der Pol [150] will be given. More complicated problems (some of them are pointed out in Section 4) are likely to require application, and possibly modification, of algorithms evolved in the fundamental investigations of Bogolyubov [22, 102], Mitropolskii [127d], and Samoilenko [24].

It is possible to outline a general approach to the problem of energy transfer (Sections 1-3). The first step of the solution is based on the mathematical theory of parametric resonance, which is used to find the initial (usually trivial) periodic mode and to determine its instability regions in the space of the system's parameters. This approach has already been employed in Chapter II, Subsection 1.6, II, 2.3, II, 2.4, and II, 2.5. The second step of the solution consists in determining periodic modes that appear at critical values of the parameters and are different from the initial mode. This step is based on the transformations given in Chapter I, Subsection 1.2 and consists in determining periodic solutions by means of the Poincaré method [188a] as applied to the transformed system. Other methods of small parameter can also be applied to the transformed system, for example, the method of averaging, which enables us to realize the third step of the solution, namely, an analysis of the transient process, which is often referred to as energy transfer. The three steps of the solution are illustrated for mechanical systems (Sections 1, 3) and for a physical system (Section 2) with two degrees of freedom.

§ 1. Loss of Stability of Vertical Vibrations of a Spring-Loaded Pendulum

For certain values of the parameters, vertical vibrations of a point mass suspended from a spring become unstable as the result of any

small transverse perturbations of motion (see, for instance [320a] and [394]). This section presents a mathematical treatment of this process.

1.1. Step 1. We consider the motion of mass m suspended from a weightless spring of unstressed length l and with force constant c (see Fig. 3). Hook's law is assumed valid. Let x and $y' = l + \lambda + y$ be the Cartesian coordinates of mass m with respect to the suspension point O, where $\lambda = mg/c$ is the static elongation of the spring. We choose the arbitrary constant in the potential energy Π of the force of gravity and the elastic force of the spring such that Π vanishes at the static equilibrium position $x = y = 0$, so that

$$K = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2),$$

$$\Pi = -mgy + \frac{1}{2} c \left[\sqrt{x^2 + (l + \lambda + y)^2} - l\right]^2 - \frac{1}{2} c\lambda^2. \qquad (1.1)$$

We denote by $\omega = \sqrt{c/m}$ the cyclic frequency of vertical vibrations of the spring-suspended mass and introduce dimensionless time $\tau = \omega t$ and coordinates $\xi = x/l$ and $\eta = y/l$. The equations of motion then become

$$\frac{d^2\eta}{d\tau^2} + \eta = \frac{1+\gamma+\eta}{\sqrt{\xi^2 + (1+\gamma+\eta)^2}} - 1 \quad \left(\gamma = \frac{\lambda}{l}\right),$$

$$\frac{d^2\xi}{d\tau^2} + \frac{\gamma}{1+\gamma}\xi = \left[\frac{1}{\sqrt{\xi^2 + (1+\gamma+\eta)^2}} - \frac{1}{1+\gamma}\right]\xi, \qquad (1.2)$$

where the expansions of the right-hand sides in some neighbourhood of $\xi = \eta = 0$ begin with second-order terms. System (1.2) has no small parameters; at the same time condition (b) of the Lyapunov theorem (I, 1.1) is violated, so that the Lyapunov method of determining periodic solutions is inapplicable. Since the system is conservative and the constraints do not depend explicitly on time, there exists the energy integral

$$\frac{1}{cl} \sqrt{2(K + \Pi)c} = \mu = \text{const} \qquad (1.3)$$

By using this equation and the Lyapunov substitution (I, 1, 2.3)

$$\eta = \rho \sin \vartheta, \quad \frac{d\eta}{d\tau} = \rho \cos \vartheta, \quad \xi = \rho z_1, \quad \frac{d\xi}{d\tau} = \rho z_2 \qquad (1.4)$$

we can reduce the order of system (1.2) by two. General formulas for transforming systems of second-order equations were given in Chap-

ter I, Subsection 1.2. Formulas (I, 1, 2.6) reduce system (1.2) to

$$W(z)=\frac{\gamma}{1+\gamma}z_1^2+z_2^2, \qquad R=-\frac{\cos\vartheta}{2(1+\gamma)^2}z_1^2+O(\rho),$$

$$\Theta=\frac{\sin\vartheta}{2(1+\gamma)^2}z_1^2+O(\rho), \quad Z_1=\frac{\cos\vartheta}{2(1+\gamma)^2}z_1^3+O(\rho),$$

$$Z_2=-\frac{\sin\vartheta}{(1+\gamma)^2}z_1+\frac{\cos\vartheta}{2(1+\gamma)^2}z_1^2z_2+O(\rho). \tag{1.5}$$

Equations (I, 1, 2.9) become ($\zeta\equiv z_1$, $\zeta'\equiv d\zeta/d\vartheta$)

$$\frac{d^2\zeta}{d\vartheta^2}+\frac{\gamma}{1+\gamma}\zeta=-\mu(1+\gamma)^{-2}\Big[\Big(1+\frac{\gamma}{1+\gamma}\zeta^2+\zeta'^2\Big)^{1/2}\sin\vartheta$$
$$+\Big(1+\frac{\gamma}{1+\gamma}\zeta^2+\zeta'^2\Big)^{-1/2}$$
$$\times\Big(\frac{1}{2}\frac{1-3\gamma}{1+\gamma}\zeta\sin\vartheta-\frac{3}{2}\zeta'\cos\vartheta\Big)\zeta\Big]\zeta+O(\mu^2). \tag{1.6}$$

The trivial solution of equation (1.6), $\xi\equiv 0$, corresponds to the vertical vibrations of mass m on a spring

$$y=Y\cos\omega(t-t_0) \tag{1.7}$$

with period $T_{\text{vert}}=2\pi/\omega$.

In positions of maximum elongation or contraction the potential energy Π of the mass is equal, according to (1.1), to $\frac{1}{2}cY^2$. Using the energy integral (1.3), we obtain for the amplitude of vertical vibrations the expression

$$Y=l\mu. \tag{1.8}$$

The stability of (1.7) was analyzed in Chapter II, Subsection 1.6 by the methods of the theory of parametric resonance. The only instability region in the $\mu\gamma$ plane is determined by inequality (II, 1, 6.5).

1.2. Step 2. We wish to determine the periodic solutions of equation (1.6) (and also of system (1.2) by virtue of substitution (1.4)) distinct from the trivial solution.

To achieve this, let us use the method of small parameter for nonautonomous systems with one degree of freedom in the form suggested by Poincaré ([188a], vol. I, Ch. III). We seek this solution in the form of the series (see I, 2.1)

$$\zeta(\vartheta)=\zeta_0(\vartheta)+\mu\zeta_1(\vartheta)+\mu^2\zeta_2(\vartheta)+\dots.$$

Substituting this series into equation (1.6), we obtain the equations for ζ_0 and ζ_1

$$\frac{d^2\zeta_0}{d\vartheta^2}+\frac{\gamma}{1+\gamma}\zeta_0=0, \tag{2.1}$$

$$\frac{d^2\zeta_1}{d\vartheta^2}+\frac{\gamma}{1+\gamma}\zeta_1=-\frac{1}{(1+\gamma)^2}\left[\left(1+\frac{\gamma}{1+\gamma}\zeta_0^2+\zeta_0'^2\right)^{1/2}\sin\vartheta\right.$$
$$\left.+\left(1+\frac{\gamma}{1+\gamma}\zeta_0^2+\zeta_0'^2\right)^{-1/2}\left(\frac{1}{2}\frac{1-3\gamma}{1+\gamma}\zeta_0\sin\vartheta-\frac{3}{2}\zeta_0'\cos\vartheta\right)\zeta_0\right]\zeta_0. \tag{2.2}$$

Equation (2.1) possesses a family of $T(\gamma)$-periodic solutions

$$\zeta_0=M_0\cos\sqrt{\frac{\gamma}{1+\gamma}}\,\vartheta+N_0\sin\sqrt{\frac{\gamma}{1+\gamma}}\,\vartheta \tag{2.3}$$
$$\left(T(\gamma)=2\pi\sqrt{\frac{1+\gamma}{\gamma}}\right).$$

Solution (2.3) can also be considered $qT(\gamma)$-periodic, where q is any natural number. Equation (1.6) is an explicit function of independent variable ϑ, and this function can also be considered $2p\pi$-periodic (p is any natural number). Therefore, solution (2.3) is generating for a $2p\pi$-periodic solution of (1.6) if and only if

$$qT(\gamma)=2p\pi \quad \text{or} \quad \gamma=\frac{1}{\left(\frac{p}{q}\right)^2-1}, \tag{2.4}$$

where p/q is an arbitrary irreducible improper fraction.

Therefore, equation (1.6) possesses periodic solutions with minimum period $2p\pi$ ($p = 2, 3, \ldots$) only for the values of relative elongation $\gamma = \lambda/l$ given by formula (2.4). Any positive rational number is either expressed by (2.4) or can be given by this formula to any degree of accuracy.

Taking into account equations (2.3) and (2.4), we recast equation (2.2) in the form

$$\frac{d^2\zeta_1}{d\vartheta^2}+\frac{q^2}{p^2}\zeta_1=-\frac{(p^2-q^2)^2}{p^4}\left(M_0\cos\frac{q}{p}\vartheta+N_0\sin\frac{q}{p}\vartheta\right)$$
$$\times\left\{\left[1+\frac{q^2}{p^2}(M_0^2+N_0^2)\right]^{1/2}\sin\vartheta+\left[1+\frac{q^2}{p^2}(M_0^2+N_0^2)\right]^{-1/2}\right.$$
$$\times\left[\frac{p^2-4q^2}{2p^2}\left(\frac{M_0^2+N_0^2}{2}+\frac{M_0^2-N_0^2}{2}\cos\frac{2q}{p}\vartheta+M_0N_0\sin\frac{2q}{p}\vartheta\right)\right.$$
$$\left.\left.\times\sin\vartheta-\frac{3}{2}\frac{q}{p}\left(M_0N_0\cos\frac{2q}{p}\vartheta-\frac{M_0^2-N_0^2}{2}\sin\frac{2q}{p}\vartheta\right)\cos\vartheta\right]\right\}. \tag{2.5}$$

This is the equation determining the first correction in μ of the $2p\pi$-periodic solution of equation (2.2) ($p = 2, 3, \ldots$).

The nonhomogeneous part of equation (2.5) contains trigonometric functions of ϑ with cyclic frequencies

$$\text{(a)}\ \frac{p-q}{p};\quad \text{(b)}\ \frac{|p-3q|}{p};\quad \text{(c)}\ \frac{p+q}{p},\ \frac{p+3q}{p}.$$

Let us find out when one of these frequencies coincides with the cyclic frequency of the generating solution

$$\text{(a)}\ \frac{p-q}{p}=\frac{q}{p}\qquad p=2,\quad q=1,\quad \gamma=\frac{1}{3};$$

$$\text{(b)}\ \frac{p-3q}{p}=\frac{q}{p},\quad p=4,\quad q=1,\quad \gamma=\frac{1}{15};$$

$$\frac{3q-p}{p}=\frac{q}{p},\quad p=2,\quad q=1,\quad \gamma=\frac{1}{3}.$$

In case (c), equality is impossible. In cases (a) and (b), equation (2.5) has $2p\pi$-periodic solutions for the listed values of p, but only for those values of M_0 and N_0 for which the terms containing $\sin(q\vartheta/p)$ and $\cos(q\vartheta/p)$ in equation (2.5) vanish. The equations for "generating amplitudes" for $\gamma = \frac{1}{3}$ $(p = 2,\ q = 1)$ are

$$N_0(4-2M_0^2+N_0^2)=0,\quad M_0(4+M_0^2-2N_0^2)=0$$

and give nonzero solutions: $M_0 = \pm 2$ and $N_0 = \pm 2$. Formula (2.3) then yields

$$\zeta_0=\pm 2\sqrt{2}\cos\left(\frac{1}{2}\vartheta\mp\frac{1}{4}\pi\right), \tag{2.6}$$

that is, a unique generating amplitude equal to $2\sqrt{2}$ for four values of the generating initial phase.

The equations for generating amplitudes become identities for $\gamma = \frac{1}{15}$. Therefore, for the remaining values of γ in (2.4) with the exception of $\gamma = \frac{1}{3}$, formula (2.3) gives a family of generating solutions of equation (1.6) involving two parameters.

Finally, we shall analyze case (a) in which equation (1.6) has a periodic solution with minimum period in ϑ equal to 4π. Formulas (I, 1, 2.4), (I, 1, 2.7), (1.5), and (2.6) yield

$$\rho=\frac{1}{\sqrt{3}}\mu+O(\mu^2),\qquad \frac{d\vartheta}{d\tau}=1+\frac{3\sqrt{3}}{8}\mu(1\pm\sin\vartheta)\sin\vartheta+O(\mu^2),$$

$$t=\frac{1}{\omega}\int_{\vartheta_0}^{\vartheta}\left[1-\frac{3\sqrt{3}}{8}\mu(1\pm\sin\vartheta)\sin\vartheta+O(\mu^2)\right]d\vartheta. \tag{2.7}$$

This gives the period of transverse vibrations of a mass suspended from a spring with the relative static elongation $\gamma = \frac{1}{3}$

$$T=\frac{4\pi}{\omega}\left[1 \mp \frac{3\sqrt{3}}{16}\mu+O(\mu^2)\right]=2T_{\text{vert}}\left[1 \mp \frac{3\sqrt{3}}{16}\mu+O(\mu^2)\right],$$

and the mass coordinates

$$x=\pm\frac{2}{3}\sqrt{6}\,l\mu\cos\left(\frac{1}{2}\omega t \mp \frac{\pi}{4}\right)+O(\mu^2),$$

$$y=\frac{1}{3}\sqrt{3}\,l\mu\sin\omega t+O(\mu^2). \tag{2.8}$$

Note that the swing period of a rigid pendulum of length $\frac{4}{3}l\left(\gamma=\frac{1}{3}\right)$ and with an amplitude given by the first formula of (2.8) is

$$T_{\text{rig}}=2\pi\sqrt{\frac{4l}{3g}}\,[1+O(\mu^2)]=2T_{\text{vert}}\,[1+O(\mu^2)].$$

It should also be noted that the second formula of (2.8) demonstrates that vertical vibrations of a swinging spring-loaded pendulum take up $\frac{1}{3}$ $[1 + O(\mu^2)]$ of the total energy of motion.

1.3. Step 3. Next we study the loss of stability of vertical vibrations. Since $\gamma = \frac{1}{3}$ is the unique value of γ for which vertical vibrations are unstable for any small value of the dimensionless amplitude $\mu = Y/l$ and since $\gamma = \frac{1}{3}$ corresponds to periodic motion (2.8), it is logical to substitute $\gamma=\frac{1}{3}$ into equation (1.6) and investigate its solutions for sufficiently small initial values of $\zeta(0)$ and $\zeta'(0)$. We consider, therefore, the equation

$$\frac{d^2\zeta}{d\vartheta^2}+\frac{1}{4}\zeta=\frac{9}{16}\mu\zeta\left\{\frac{3}{2}\zeta\frac{d\zeta}{d\vartheta}\cos\vartheta\left[1+\frac{1}{4}\zeta^2+\left(\frac{d\zeta}{d\vartheta}\right)^2\right]^{-1/2}\right.$$
$$\left.-\left[1+\frac{1}{4}\zeta^2+\left(\frac{d\zeta}{d\vartheta}\right)^2\right]^{1/2}\sin\vartheta\right\}+O(\mu^2). \tag{3.1}$$

The Van der Pol substitution [150]

$$\zeta=a\cos\left(\frac{1}{2}\vartheta+\varphi\right),\quad \frac{d\zeta}{d\vartheta}=-\frac{1}{2}a\sin\left(\frac{1}{2}\vartheta+\varphi\right) \tag{3.2}$$

reduces equation (3.1) to an equivalent system with respect to the

slowly changing variables a and φ

$$\frac{da}{d\vartheta}=\frac{9}{16}\mu a\left[\sqrt{1+\frac{1}{4}a^2}\sin\vartheta\right.$$
$$\left.-\frac{3}{8}a^2\left(1+\frac{1}{4}a^2\right)^{-1/2}\sin(\vartheta+2\varphi)\cos\vartheta\right]\sin(\vartheta+2\varphi)+O(\mu^2),$$

$$\frac{d\varphi}{d\vartheta}=\frac{9}{16}\mu\left[\sqrt{1+\frac{1}{4}a^2}\sin\vartheta\right.$$
$$\left.-\frac{3}{8}a^2\left(1+\frac{1}{4}a^2\right)^{-1/2}\sin(\vartheta+2\varphi)\cos\vartheta\right][1+\cos(\vartheta+2\varphi)]+O(\mu^2).$$

Averaging over the independent variable ϑ results in the truncated Van der Pol equations

$$\frac{da}{d\vartheta}=\frac{9}{64}\mu a\sqrt{4+a^2}\cos 2\varphi+O(\mu^2),$$
$$\frac{d\varphi}{d\vartheta}=\frac{9}{128}\mu\frac{a^2-8}{\sqrt{4+a^2}}\sin 2\varphi+O(\mu^2). \tag{3.3}$$

Dividing the first equation of (3.3) by the second and integrating, we obtain

$$\frac{1}{2}\int_{a_0}^{a}\frac{a^2-8}{a(4+a^2)}\,da=\int_{\varphi_0}^{\varphi}\cot 2\varphi\,d\varphi+O(\mu).$$

Completing the integration, we find the first integral of the truncated equations (3.3):

$$a^4\sin^2 2\varphi=c^2(4+a^2)^3+O(\mu)\quad\left(c^2=\frac{a_0^4}{(4+a_0^2)^3}\sin^2 2\varphi_0\right).$$

Unfortunately, the substitution of this integral into the first equation of (3.3) results in unmanageable quadratures. Of course, the truncated Van der Pol equations give only the first approximation of the solution to (3.1) (higher-order approximations are obtained by the Picard method of successive approximations, with the estimates of accuracy the method provides). Consequently, we limit our analysis to an approximate integration of system (3.3). The second equation of (3.3) shows that $d\varphi/d\vartheta\leqslant 0$ for $|a|\leqslant 2\sqrt{2}$ and $|\varphi_0|>|\varphi|\geqslant 0$. By (3.2), $|\zeta|\leqslant 2\sqrt{2}$ for $|a|\leqslant 2\sqrt{2}$ (we recall that $2\sqrt{2}$ is the amplitude of the generating solution (2.6)); therefore, we can assume $\cos 2\varphi\approx 1$ for sufficiently small φ_0 over the whole duration of the loss of stability process (transition from vertical vibrations to pendulum swinging). The first equation of (3.3)

then yields for $a_0 = a(0) > 0$

$$\int_{a_0}^{a} \frac{da}{a\sqrt{4+a^2}} = \frac{9}{64}\mu\vartheta,$$

which after integration gives

$$\ln\left(\frac{\sqrt{4+a^2}-2}{a}\,\frac{a_0}{\sqrt{4+a_0^2}-2}\right) = \frac{9}{32}\mu\vartheta.$$

From this we derive an approximate expression for the amplitude as a function of ϑ

$$a = \frac{4b_0 \exp\left(\frac{9}{32}\mu\vartheta\right)}{1-b_0^2\exp\left(\frac{9}{16}\mu\vartheta\right)} \quad \left(b_0 = \frac{1}{a_0}[\sqrt{4+a_0^2}-2]\right). \tag{3.4}$$

Let us calculate the time of transition from vertical vibrations ($\zeta \equiv 0$) to transverse vibrations (2.6). Setting $a = 2\sqrt{2}$ in (3.4), we obtain the corresponding value of $\tilde{\vartheta}$

$$\tilde{\vartheta} = \frac{32}{9\mu}\ln\left(\frac{\sqrt{3}-1}{\sqrt{2}}\,\frac{a_0}{\sqrt{4+a_0^2}-2}\right) \tag{3.5}$$

Note that the transient process covers a time interval of the order of $O(\mu^{-1})$, which corresponds to the algorithm of asymptotic integration in the method of averaging ([129], Ch. III, § 4, Subsec. 4).

Since φ_0 is small, $a_0 \approx \zeta(0)$ and for small a_0 (3.5) becomes

$$\tilde{\vartheta} = \frac{32}{9\mu}\ln\frac{2\sqrt{2}(\sqrt{3}-1)}{\zeta(0)} \tag{3.6}$$

Formulas (1.4) enable us to express $\zeta(0)$ in terms of the initial values of the variables, namely,

$$(0) = \frac{x(0)}{l\rho(0)}, \quad \rho = \sqrt{\eta^2 + \left(\frac{d\eta}{d\tau}\right)^2}$$

$$\zeta(0) = x(0)\left[y(0)^2 + \omega^{-2}y(\omega)^2\right]^{-1/2}$$

Finally, the last formula of (2.7) gives the sought time of transition from vertical vibrations to transverse (swinging) vibrations (2.8)

$$\tilde{t} = \frac{1}{\omega}\left[\left(1 \mp \frac{3\sqrt{3}}{16}\mu\right)\tilde{\vartheta} + O(\mu)\right] \tag{3.7}$$

We recall that μ can be determined by (1.3)

$$\mu = \frac{1}{cl}\sqrt{2(K_0+\Pi_0)c}$$

or, if the transverse perturbation $x(0)$ is assumed small, by (1.8). The two signs in (3.7) are related to the phase of the pendulum swings in (2.6).

§ 2. On Coupling of Radial and Vertical Oscillations of Particles in Cyclic Accelerators

In this section we define pure radial (side-to-side) oscillations and analyze their stability. Oscillation equations are then transformed and vertical radial (up-and-down) oscillations are found by means of the Poincaré method of small parameter. There exists a unique value of the principal physical parameter for which the reduced amplitude of up-and-down oscillations is unique, whereas pure side-to-side oscillations of any amplitude, no matter how small, are unstable at this value of the parameter. Then we describe the transient process and find the time of transition from pure side-to-side to up-and-down oscillations and point out an analogy with vibrations of a spring-loaded pendulum in transient processes.

2.1. Step 1. The equations of betatron oscillations of particles in cyclic accelerators with weak focusing are given by ([246], (4.6))

$$\ddot{\xi}+\omega^2(1-n)\xi=-\frac{1}{2}k\eta^2,\quad \ddot{\eta}+\omega^2 n\eta=-k\xi\eta. \tag{1.1}$$

Here $\xi=(r-r_0)/r_0$, $\eta=z/r_0$; r and z are two of the cylindrical coordinates of a particle; a dot above a symbol denotes differentiation with respect to time; and the constants ω, n, and k are

$$\omega=\frac{eH(r_0)}{mc},\quad n=-\frac{r_0}{H(r_0)}\left(\frac{\partial H}{\partial r}\right)_0\quad (0<n<1),$$
$$k=-\frac{r_0}{H(r_0)}\left(\frac{\partial^2 H}{\partial r^2}\right)_0, \tag{1.2}$$

where m and e are the mass and charge of a particle; c is the speed of light; $H_z=H(r)$ is the vertical component of the magnetic field intensity vector; and partial derivatives are evaluated at $r=r_0$, that is, at the path radius corresponding to the fixed energy of the particles.

We introduce dimensionless time $\tau=\omega\sqrt{1-n}\,t$ and recast system (1.1) in the form

$$\frac{d^2\xi}{d\tau^2}+\xi=-\frac{1}{2}\beta\eta^2,\quad \frac{d^2\eta}{d\tau^2}+\alpha\eta=-\beta\xi\eta, \tag{1.3}$$

where the positive parameters α and β are

$$\alpha=\frac{n}{1-n}\qquad \beta=\frac{k}{\omega^2(1-n)}. \tag{1.4}$$

Integral (I, 1, 2.2) for system (1.3) is the integral of particle energy corresponding to the oscillatory part of the motion

$$\mathscr{H}=\xi^2+\left(\frac{d\xi}{d\tau}\right)^2+W+S_3=\mu^2$$
$$\left(W=\alpha\eta^2+\left(\frac{d\eta}{d\tau}\right)^2,\quad S_3=\beta\xi\eta^2\right). \tag{1.5}$$

Before transforming system (1.3), we note that it has a solution (pure side-to-side oscillations)

$$\eta\equiv 0,\quad \xi=\mu\cos(\tau-\tau_0). \tag{1.6}$$

As follows from integral (1.5), the constant μ^2 is equal to the squared amplitude of pure side-to-side oscillations. To analyze the stability of these oscillations, we set in (1.3)

$$\xi=\mu\cos(\tau-\tau_0)+x,\quad \eta=0+y,$$

which yields the variational equations

$$\frac{d^2x}{d\tau^2}+x=0,\quad \frac{d^2y}{d\tau^2}+[\alpha+\beta\mu\cos(\tau-\tau_0)]\,y=0.$$

Hence, the instability of side-to-side oscillations (1.6) is determined by the instability of the trivial solution of the second equation (Mathieu's equation). The principal physical parameter is n (see (1.2)). The instability regions in the μn plane are contiguous to the critical points on the n axis given by the equalities (see [80], § VII, 1)

$$2\sqrt{\alpha(n)}=l\qquad (l=1,\ 2,\ \ldots),$$

whence

$$n=n_l=\frac{l^2}{4+l^2}\quad (l=1,\ 2,\ \ldots).$$

A wide instability range (that is, with the angle between the tangents distinct from zero) is contiguous only to point $n_1=\frac{1}{5}$, and the slope of the tangents is given by (II, 2, 5.6)

$$\chi^{\mp}=\mp\left.\frac{\frac{1}{2}\beta}{\frac{d\alpha}{dn}}\right|_{n=n_1}=\mp\frac{k}{2\omega^2}(1-n_1)=\mp\frac{2}{5}\,\frac{k}{\omega^2}.$$

Therefore, in the first approximation the only wide instability region (1.6) in the μn plane is determined by the inequalities

$$\frac{1}{5}-\frac{2}{5}\,\frac{k}{\omega^2}\mu+\ldots<n<\frac{1}{5}+\frac{2}{5}\,\frac{k}{\omega^2}\mu+\ldots \tag{1.7}$$

with the critical value $n=n_1=\frac{1}{5}$.

2.2. Step 2. In order to transform system (1.3), from formulas (II, 1, 2.6) we find

$$R(\rho,\ \vartheta,\ z_1,\ z_2) = -\frac{1}{2}\beta z_1^2 \cos\vartheta,\quad \Theta(\rho,\ \vartheta,\ z_1,\ z_2) = \frac{1}{2}\beta z_1^2 \sin\vartheta,$$

$$Z_1(\rho,\ \vartheta,\ z_1,\ z_2) = \frac{1}{2}\beta z_1^3 \cos\vartheta,$$

$$Z_2(\rho,\ \vartheta,\ z_1,\ z_2) = -\beta z_1 \sin\vartheta + \frac{1}{2}\beta z_1^2 z_2 \cos\vartheta. \tag{2.1}$$

Equation (I, 1, 2.9) now becomes ($z_1 \equiv \zeta$, $\zeta' \equiv d\zeta/d\vartheta$)

$$\frac{d^2\zeta}{d\vartheta^2} + \alpha\zeta = -\mu\beta\zeta\Big[\sqrt{1+\alpha\zeta^2+\zeta'^2}\sin\vartheta$$

$$+(1+\alpha\zeta^2+\zeta'^2)^{-1/2}\left(\frac{1-4\alpha}{2}\zeta\sin\vartheta - \frac{3}{2}\zeta'\cos\vartheta\right)\zeta\Big] + O(\mu^2). \tag{2.2}$$

For $\alpha = \gamma/(1+\gamma)$ and $\beta = (1+\gamma)^{-2}$, equation (2.2) becomes identical to (1, 1.6) and demonstrates an analogy with Section 1. By using this analogy, we can show that equation (2.2) has periodic solutions with minimum period $2p\pi$ ($p = 1, 2, \ldots$) only for values $\alpha = q^2/p^2$, or, by virtue of (1.4), for

$$n = \frac{q^2}{p^2+q^2} \tag{2.3}$$

where q and p are any mutually prime numbers. The set of values of n given by (2.3) is dense in the interval (0, 1) of variation of n; in other words, each $n \in (0, 1)$ is either given by (2.3) or can be expressed by means of (2.3) to any desired degree of accuracy. Formula (1, 2.3) yields a family of generating solutions to equation (2.2) in two parameters for all n in (2.3) with the exception of $n = \frac{1}{5}$ ($q = 1$ and $p = 2$).

Finally, let us analyze the case of $n = \frac{1}{5}$, that is, of the unique value of n for which a generating solution of equation (2.2) has the form (1, 2.6) and for which pure side-to-side oscillations (1.6) are unstable for any amplitude μ no matter how small (see (1.7)). Analogously to (1, 2.7), we obtain

$$\rho = \frac{1}{3}\sqrt{3}\,\mu + O(\mu^2),$$

$$\frac{d\vartheta}{d\tau} = 1 + \frac{2}{3}\sqrt{3}\,\beta\mu(1 \pm \sin\vartheta)\sin\vartheta + O(\mu^2).$$

This yields for the period of up-and-down oscillations for $n=\frac{1}{5}$

$$T=\frac{1}{\omega\sqrt{1-\frac{1}{5}}}\int_0^{4\pi}\left[1+\frac{2}{3}\sqrt{3}\beta\mu(1\pm\sin\vartheta)\sin\vartheta+O(\mu^2)\right]^{-1}d\vartheta$$

$$=\frac{2\sqrt{5}\pi}{\omega}\left[1\mp\frac{5}{12}\sqrt{3}\frac{k}{\omega^2}\mu+O(\mu^2)\right], \qquad (2.4)$$

and for the expression of motion, analogously to (1, 2.8),

$$\xi=\frac{1}{3}\sqrt{3}\mu\sin\left(\frac{2}{5}\sqrt{5}\,\omega t\right)+O(\mu^2),$$

$$\eta=\rho\zeta=\pm\frac{2}{3}\sqrt{6}\mu\cos\left(\frac{\sqrt{5}}{5}\omega t\mp\frac{\pi}{4}\right)+O(\mu^2). \qquad (2.5)$$

The value of μ is determined by the initial value of the reduced energy (1.5) of the oscillations.

2.3. Step 3. We wish to describe the process of energy transfer for $n=\frac{1}{5}$, that is, the process of transition from unstable pure side-to-side oscillations (1.6) to up-and-down oscillations (2.5). The truncated Van der Pol equations take the form of (1, 3.3), with multipliers $\frac{9}{64}$ and $\frac{9}{128}$ replaced by $\frac{1}{4}\beta$ and $\frac{1}{8}\beta$, respectively. The expression for the Van der Pol amplitude is found in a manner similar to that of Subsection 1.3

$$a=\frac{4b_0\exp\left(\frac{1}{2}\beta\mu\vartheta\right)}{1-b_0^2\exp(\beta\mu\vartheta)}\quad\left(b_0=\frac{1}{a_0}\left[\sqrt{4+a_0^2}-2\right]\right). \qquad (2.6)$$

The duration $\tilde{t}$ of the process of transition from pure side-to-side oscillations (1.6) to up-and-down oscillations (2.5) for $n=\frac{1}{5}$ is determined by the formulas

$$\tilde{t}=\frac{\sqrt{5}}{2\omega}\left[\left(1\mp\frac{1}{3}\sqrt{3}\tilde{\beta}\mu\right)\tilde{\vartheta}+O(\mu)\right], \qquad (2.7)$$

where

$$\tilde{\beta}=\frac{5}{4}\frac{k}{\omega^2},\quad \tilde{\vartheta}=\frac{2}{\tilde{\beta}\mu}\ln\frac{2\sqrt{2}(\sqrt{3}-1)}{\zeta(0)},$$

$$\zeta(0)=\eta(0)\left[\xi^2(0)+\frac{5}{4}\omega^{-2}\dot{\xi}^2(0)\right]^{-1/2},$$

and μ is determined by the initial value of the reduced energy (1.5) of the oscillations.

§ 3. Loss of Stability of Vertical Oscillations of a Pendulum Subject to Elastic Guided Suspension

Step 1 of the solution was given in Chapter II, Subsection 2.5. The initial periodic mode is a vertical oscillation (II, 2, 5.4).

3.1. Determination of nontrivial periodic modes (Step 2). By employing formulas (II, 2, 5.1) and (II, 2, 5.3), we transform the energy integral to the form

$$\left(\frac{d\eta}{d\tau}\right)^2+\eta^2+\alpha\left(\frac{d\varphi}{d\tau}\right)^2+2\alpha\gamma(1-\cos\varphi$$
$$-2\alpha\sin\varphi\frac{d\eta}{d\tau}\frac{d\varphi}{d\tau}=\mu^2. \qquad (1.1)$$

Expanding the right-hand sides of the system of equations (II, 2, 5.2) and the left-hand side of the energy integral in power series, we obtain

$$\frac{d^2\eta}{d\tau^2}+\eta=\alpha\left(\frac{d\varphi}{d\tau}\right)^2-\alpha\gamma\varphi^2+(3),\qquad \frac{d^2\varphi}{d\tau^2}+\gamma\varphi=-\eta\varphi+(3),$$
$$\left(\frac{d\eta}{d\tau}\right)^2+\eta^2+\alpha\gamma\varphi^2+\alpha\left(\frac{d\varphi}{d\tau}\right)^2+(3)=\mu^2.$$

Comparing this with (I, 1, 2.1) and (I, 1, 2.2), on the basis of formulas (I, 1, 2.6), we find

$$R=\alpha(-\gamma z_1^2+z_2^2)\cos\vartheta+O(\rho),$$
$$\Theta=\alpha(\gamma z_1^2-z_2^2)\sin\vartheta+O(\rho),$$
$$Z_1=\alpha(\gamma z_1^3-z_1z_2^2)\cos\vartheta+O(\rho),$$
$$Z_2=-z_1\sin\vartheta+\alpha(\gamma z_1^2z_2-z_2^3)\cos\vartheta+O(\rho).$$

In the case under discussion, system (I, 1, 2.9) is reduced to a single equation ($z_1\equiv\zeta$, $\zeta'\equiv d\zeta/d\vartheta$)

$$\frac{d^2\zeta}{d\vartheta^2}+\gamma\zeta=\mu\{-\zeta\sin\vartheta\sqrt{1+\alpha(\gamma\zeta^2+\zeta'^2)}$$
$$+\alpha[1+\alpha(\gamma\zeta^2+\zeta'^2)]^{-1/2}[2\zeta\sin\vartheta(\gamma^2\zeta^2+\zeta'^2-3\gamma\zeta'^2)$$
$$+\zeta'\cos\vartheta(5\gamma\zeta^2-\zeta'^2)]\}+O(\mu^2). \qquad (1.2)$$

By calculations similar to those of Subsections 1.2, 2.2 and by using the Poincaré method [188a], we arrive at a generating periodic solution for the critical value $\gamma=\frac{1}{4}$

$$\zeta_0=\pm2\sqrt{\frac{2}{\alpha}}\cos\left(\frac{1}{2}\vartheta\mp\frac{1}{4}\pi\right). \qquad (1.3)$$

Together with formulas (I, 1, 2.7), (I, 1, 2.3), and (II, 2, 5.3), the above equation yields a periodic solution distinct from vertical

vibrations, namely, for transverse "pendulum" vibrations

$$y=\frac{1}{3}\sqrt{3}\,l_2\mu\sin\sqrt{\frac{c}{m_1+m_2}}\,t+O(\mu^2),$$

$$\varphi=\pm\frac{2}{3}\sqrt{\frac{6}{\alpha}}\,\mu\cos\left(\frac{1}{2}\sqrt{\frac{c}{m_1+m_2}}\,t\mp\frac{\pi}{4}\right)+O(\mu^2). \qquad (1.4)$$

3.2. Transient process (Step 3). The Van der Pol substitution [150]

$$\zeta=\frac{a}{\sqrt{\alpha}}\cos\left(\frac{1}{2}\vartheta+\psi\right),\quad \frac{d\zeta}{d\vartheta}=-\frac{1}{2}\frac{a}{\sqrt{\alpha}}\sin\left(\frac{1}{2}\vartheta+\psi\right) \qquad (2.1)$$

reduces equation (1.2) for $\gamma=\frac{1}{4}$ to an equivalent system with respect to slowly changing variables a and ψ

$$\frac{da}{d\vartheta}=\mu a\left\{\sqrt{1+\frac{1}{4}a^2}\sin\vartheta\sin(\vartheta+2\psi)\right.$$
$$-\left(1+\frac{1}{4}a^2\right)^{-1/2}\left[\frac{1}{8}a^2\sin\vartheta\sin(\vartheta+2\psi)-\frac{1}{4}a^2\cos\vartheta\right.$$
$$\left.\left.-\frac{1}{8}a^2\cos\vartheta\cos(\vartheta+2\psi)+\frac{3}{8}a^2\cos\vartheta\cos^2(\vartheta+2\psi)\right]\right\}+O(\mu^2),$$

$$\frac{d\psi}{d\vartheta}=\mu\left\{\sqrt{1+\frac{1}{4}a^2}\sin\vartheta\,[1+\cos(\vartheta+2\psi)]\right.$$
$$-\left(1+\frac{1}{4}a^2\right)^{-1/2}\left[\frac{1}{8}a^2\sin\vartheta\,(1+\cos(\vartheta+2\psi))\right.$$
$$\left.\left.-\frac{1}{4}a^2\cos\vartheta\sin(\vartheta+2\psi)-\frac{3}{16}a^2\cos\vartheta\sin(2\vartheta+4\psi)\right]\right\}+O(\mu^2). \qquad (2.2)$$

The truncated Van der Pol equations obtained by averaging over the independent variable explicitly included in (2.2) are

$$\frac{da}{d\vartheta}=\frac{1}{4}\mu a\sqrt{4+a^2}\cos 2\psi+O(\mu^2),$$

$$\frac{d\psi}{d\vartheta}=\frac{1}{8}\mu\frac{a^2-8}{\sqrt{4+a^2}}\sin 2\psi+O(\mu^2). \qquad (2.3)$$

An attempt to solve equations (2.3) leads to unmanageable quadratures. Of course, the truncated Van der Pol equations (2.3) can give only the first approximation to the solution of system (2.2). Therefore we are satisfied with approximate integration of system (2.3). The second equation of (2.3) implies that $d\psi/d\vartheta\leqslant 0$ for $|a|\leqslant 2\sqrt{2}$, and hence $|\psi_0|>|\psi|\geqslant 0$. By (2.1), $|\zeta|\leqslant 2\sqrt{2}/\sqrt{\alpha}$ for $|a|\leqslant 2\sqrt{2}$ (we recall that $2\sqrt{2}/\sqrt{\alpha}$ is the amplitude of the generating solution (1.3)); therefore, we can assume that $\cos\psi\approx 1$ for sufficiently small ψ_0 over the whole duration of the transient

process (i.e. during the transition from vertical vibrations (II, 2, 5.4) to transverse vibrations (1.4)). The first equation of (2.3) then yields for $a_0 = a\,(0) > 0$

$$\int_{a_0}^{a} \frac{da}{a\sqrt{4+a^2}} = \frac{1}{4}\mu\vartheta.$$

After integration, we obtain

$$\ln\left(\frac{\sqrt{4+a^2}-2}{a}\,\frac{a_0}{\sqrt{4+a_0^2}-2}\right) = \frac{1}{2}\mu\vartheta.$$

This gives an approximate expression for the amplitude of the vibrations

$$a = \frac{4b_0 \exp\left(\frac{1}{2}\mu\vartheta\right)}{1-b_0^2\exp(\mu\vartheta)}, \quad \left(b_0 = \frac{1}{a_0}\left[\sqrt{4+a_0^2}-2\right]\right). \tag{2.4}$$

We now calculate the time of transition from vertical vibrations after perturbation ($a = a_0 > 0$) to transverse vibrations ($a = 2\sqrt{2}$). Assuming $a = 2\sqrt{2}$ in (2.4), we obtain for the corresponding value of $\tilde{\vartheta}$

$$\tilde{\vartheta} = \frac{2}{\mu}\ln\left(\frac{\sqrt{3}-1}{\sqrt{2}}\,\frac{a_0}{\sqrt{4+a_0^2}-2}\right).$$

Note that the transition time is of the order of $O\,(\mu^{-1})$, which corresponds to the algorithm of asymptotic integration in the method of averaging. Since ψ_0 is small, $a_0 \approx \zeta\,(0)$. For small a_0 the preceding formula gives

$$\tilde{\vartheta} = \frac{2}{\mu}\ln\frac{2\sqrt{2}(\sqrt{3}-1)}{\zeta(0)}.$$

Now $\zeta\,(0)$ can be expressed in terms of the initial values of the initial variables by means of formulas (I, 1, 2.3)

$$\zeta = \frac{\varphi}{\rho}, \quad \rho = \sqrt{\eta^2 + \left(\frac{d\eta}{d\tau}\right)^2},$$

$$\zeta(0) = \varphi(0)\left[y^2(0) + \omega^{-2}\left(\frac{dy}{dt}\right)_0^2\right]^{-1/2}.$$

The transition time (time of energy transfer) for the critical value $\gamma = \frac{1}{4}$ $\left(\text{i.e. for } \lambda = \frac{1}{4}l_2\right)$ becomes

$$\tilde{t} = \sqrt{\frac{m_1+m_2}{c}}\,\tilde{\vartheta} + O\,(1).$$

Formulas (II, 2, 5.1), (II, 2, 5.3), and (1.1) now yield

$$\mu = \frac{1}{cl_2}\sqrt{2(K_0 + \Pi_0)c},$$

or, if the transverse perturbation $\varphi(0)$ is assumed to be sufficiently small, one can use formula (II, 2, 5.4).

§ 4. Periodic Modes of a Pendulum Subject to Elastic Free Suspension

4.1. Transformation of equations of motion. We expand the right-hand sides of system (II, 2, 4.2) into power series in the respective variables and recast (II, 2, 4.2) in the form

$$\begin{gathered}\eta'' + \eta = Y(\eta, \xi, \varphi, \varphi'),\\ \xi'' + \frac{\gamma}{(1-\alpha)(1+\gamma)}\xi - \frac{\alpha\gamma}{1-\alpha}\varphi = X(\eta, \xi, \varphi, \varphi'),\\ \varphi'' - \frac{\gamma}{(1-\alpha)(1+\gamma)\beta}\xi + \frac{\gamma}{(1-\alpha)\beta}\varphi = \Phi(\eta, \xi, \varphi, \varphi'). \qquad (1.1)\end{gathered}$$

The primes denote the derivatives with respect to τ, and the expansions of the functions Y, X, and Φ begin with terms of order not lower than two. The roots of the characteristic equation of the linear part are $\mp i\omega_1$ and $\mp i\omega_2$, where

$$\omega_{1,2}^2 = \frac{\gamma}{2(1-\alpha)(1+\gamma)\beta} \times [1+\gamma+\beta \mp \sqrt{(1+\gamma+\beta)^2 - 4\beta(1-\alpha)(1+\gamma)}].$$

Since $0 < \alpha < 1$, and $\beta > 0$, $\gamma > 0$,

$$0 < (1+\gamma+\beta)^2 - 4(1-\alpha)(1+\gamma)\beta < (1+\gamma+\beta)^2$$

and, consequently, ω_1^2 and ω_2^2 are positive.

The energy integral (I, 1, 2.2) takes the form

$$\eta'^2 + \eta^2 + \xi'^2 + \frac{\gamma}{1+\gamma}\xi^2 + \alpha\beta^2\varphi'^2 + 2\alpha\beta\varphi'\xi' + S_3 = \mu^2. \qquad (1.2)$$

Substitution (I, 1, 2.3) now becomes

$$\begin{gathered}\eta' = \rho\cos\vartheta, \quad \eta = \rho\sin\vartheta,\\ \xi = \rho z_1, \quad \varphi = \rho z_2, \quad \xi' = \rho z_3, \quad \varphi' = \rho z_4,\end{gathered}$$

so that (1.2) yields

$$\rho = \mu\left(1 + \frac{\gamma}{1+\gamma}z_1^2 + z_3^2 + 2\alpha\beta z_3 z_4 + \alpha\beta^2 z_4^2\right) + O(\mu^2). \qquad (1.3)$$

System (I, 1, 2.9) is reduced, as a result, to

$$\frac{d^2 z_1}{d\vartheta^2}+\frac{\gamma}{(1-\alpha)(1+\gamma)}z_1-\frac{\alpha\gamma}{1-\alpha}z_2=O(\mu),$$
$$\frac{d^2 z_2}{d\vartheta^2}-\frac{\gamma}{\beta(1-\alpha)(1+\gamma)}z_1+\frac{\gamma}{\beta(1-\alpha)}z_2=O(\mu). \tag{1.4}$$

4.2. Periodic solutions. The general solution of the generating system (1.4) (i.e. for $\mu = 0$) is

$$z_{10}=[\gamma-\omega_1^2(1-\alpha)\beta](M_1\sin\omega_1\vartheta+M_2\cos\omega_1\vartheta)$$
$$+[\gamma-\omega_2^2(1-\alpha)\beta](M_3\sin\omega_2\vartheta+M_4\cos\omega_2\vartheta),$$

$$z_{20}=\frac{\gamma}{1+\gamma}(M_1\sin\omega_1\vartheta+M_2\cos\omega_1\vartheta+M_3\sin\omega_2\vartheta+M_4\cos\omega_2\vartheta).$$

Substituting these values into $y = l_1\rho\sin\vartheta$, $x = l_1\rho z_1$, $\varphi = \rho z_2$, and (1.3) and taking into account that $\vartheta = \tau + O(\mu)$ by virtue of the second equation of (I, 1, 2.4), we obtain the following approximation to a periodic solution of system (II, 2, 4.1)

$$y=\mu l_1 F(\tau)\sin\tau+O(\mu^2),$$
$$x=\mu l_1 F(\tau)\{[\gamma-\beta\omega_1^2(1-\alpha)](M_1\sin\omega_1\tau+M_2\cos\omega_1\tau)$$
$$+[\gamma-\omega_2^2\beta(1-\alpha)](M_3\sin\omega_2\tau+M_4\cos\omega_2\tau)\}+O(\mu^2),$$
$$\varphi=\mu\frac{\gamma}{1+\gamma}F(\tau)(M_1\sin\omega_1\tau+M_2\cos\omega_1\tau$$
$$+M_3\sin\omega_2\tau+M_4\cos\omega_2\tau)+O(\mu^2),$$

where

$$F(\tau)=\Big(1+\frac{1}{2}\gamma^2(1+\gamma)^{-2}\{\beta(1-\alpha)(1+\gamma-\beta)[\omega_1^2(M_1^2+M_2^2)$$
$$+\omega_2^2(M_3^2+M_4^2)]-\gamma(1+\gamma-\beta+2\alpha\beta)(M_1^2+M_2^2$$
$$+M_3^2+M_4^2)+\alpha\beta\gamma(M_1\sin\omega_1\tau+M_2\cos\omega_1\tau$$
$$+M_3\sin\omega_2\tau+M_4\cos\omega_2\tau)^2\}\Big)^{-1/2}.$$

Here $\tau = \sqrt{c/(m_1+m_2)}\,(t-t_0)$, and μ is determined from the initial value of the reduced energy (1.2). An analysis of the first equation in (I, 2, 1.4) (the equation for generating amplitudes) yields the critical values of the parameters α, β, and γ for which the generating amplitudes M_1, M_2, M_3, and M_4 are dependent (this corresponds to limit cycles). The periodic solution obtained, which contains six arbitrary constants M_1, M_2, M_3, M_4, t_0, and μ, is a general solution for all the remaining values of the parameters α, β, and γ for sufficiently small $\mu > 0$.

CHAPTER IV

OSCILLATIONS IN MODIFIED LYAPUNOV SYSTEMS

§ 1. Lyapunov Systems with Damping

An unperturbed nonlinear autonomous Lyapunov system of order $2k + 2$ is perturbed by an analytic damping of sufficiently small norm. The perturbed system is transformed by a method that reduces the unperturbed system to a quasilinear nonautonomous system of order $2k$. A solution of the latter system is assumed known for a sufficiently small square root (as compared with unity) of the initial reduced energy of the system. The first and all subsequent corrections of the corresponding solution of the perturbed system (i.e. with the same initial conditions) are found from a complete system of variational equations in the Poincaré parameter, that is, a nonhomogeneous system of linear differential equations of order $2k + 1$ with variable coefficients. According to Poincaré's results ([188a], vol. I, Ch. II) the integration of a system of variational equations can be reduced to quadratures if the general integral of the unperturbed system is known. Subsections 1.3, 1.4, and 1.5 are devoted to examples.

1.1. Transformation of equations of motion. We consider a class of Lyapunov systems with damping where each system is represented by a second-order equation

$$\frac{d^2u}{d\tau^2}+u-U(u,\ \dot{u},\ v_1,\ \ldots,v_k,\ \dot{v}_1,\ \ldots,\ \dot{v}_k)=-2\varepsilon F_0(\dot{u},\ \dot{v}_1,\ \ldots,\dot{v}_k),$$

$$\frac{d^2v_\varkappa}{d\tau^2}+a_{\varkappa 1}v_1+\ldots+a_{\varkappa k}v_k-V_\varkappa(u,\ \dot{u},\ v_1,\ \ldots,v_k,\ \dot{v}_1,\ \ldots,\dot{v}_k)$$

$$=-2\varepsilon F_\varkappa(\dot{u},\ \dot{v}_1,\ \ldots,\dot{v}_k),\quad (\varkappa=1,\ \ldots,k). \tag{1.1}$$

The dots denote the derivatives with respect to τ; $a_{j\varkappa} = a_{\varkappa j}$ $(\varkappa, j = 1, \ldots, k)$ are real constants; U, $V_1, \ldots, V_k$, F_0, $F_1, \ldots, F_k$ are real analytic functions of the respective variables; the expansions of F_0, $F_1, \ldots, F_k$ do not contain terms of zeroth power; the expansions of U, $V_1, \ldots, V_k$ begin with terms of power not lower than two; and, finally, $\varepsilon > 0$. We assume that the unperturbed system (1.1) (i.e. system (1.1) for $\varepsilon = 0$) possesses the first integral,

which is inevitably an analytic function of the variables u, $\dot{u}$, $v_1, \ldots$ $\ldots, v_k, \dot{v}_1, \ldots, \dot{v}_k$ ([108a], § 38; [111b], Ch. VII, § 1)

$$H = \dot{u}^2 + u^2 + W(v_1, \ldots, v_k, \dot{v}_1, \ldots, \dot{v}_k) + S_3(u, \dot{u}, v_1, \ldots, v_k, \dot{v}_1, \ldots, \dot{v}_k) = \mu^2 \quad (\mu > 0), \tag{1.2}$$

where W is a quadratic form and S_3 denotes the sum of the terms of power greater than two. We assume for the specified resistance forces $F_0, F_1, \ldots, F_k$ that their work is negative for any virtual displacement (which in the case under discussion coincides with one of the actual displacements), that is,

$$-F_0(\dot{u}, \dot{v}_1, \ldots, \dot{v}_k)\,\dot{u} - \sum_{\varkappa=1}^{k} F_\varkappa(\dot{u}, \dot{v}_1, \ldots, \dot{v}_k)\,\dot{v}_\varkappa < 0. \tag{1.3}$$

Condition (1.3) signifies that $\alpha F(\alpha) > 0$ $(\alpha \neq 0)$ in the simplest nonlinear case of $F_j = F(\dot{v}_j)$ $(j = 0, 1, \ldots, k;\ \dot{v}_0 \equiv \dot{u})$. In the linear case condition (1.3) signifies that dissipation is complete.

The Lyapunov substitution

$$\begin{aligned} u &= \rho \sin \vartheta, & \dot{u} &= \rho \cos \vartheta & &(\rho \geqslant 0), \\ v_\varkappa &= \rho z_\varkappa, & \dot{v}_\varkappa &= \rho z_{k+\varkappa} & &(\varkappa = 1, \ldots, k) \end{aligned} \tag{1.4}$$

reduces system (1.1) and the first integral (1.2) of the unperturbed system to the following form

$$\begin{aligned}
\frac{d\vartheta}{d\tau} &= 1 - \frac{1}{\rho} U(\rho \sin \vartheta, \rho \cos \vartheta, \rho z) \sin \vartheta + 2\varepsilon \frac{1}{\rho} F_0(\rho \cos \vartheta, \rho z^{(2)}) \sin \vartheta, \\
\frac{d\rho}{d\tau} &= U(\rho \sin \vartheta, \rho \cos \vartheta, \rho z) \cos \vartheta - 2\varepsilon \frac{1}{\rho} F_0(\rho \cos \vartheta, \rho z^{(2)}) \cos \vartheta, \\
\frac{dz_\varkappa}{d\tau} &= z_{k+\varkappa} - \frac{1}{\rho} z_\varkappa U(\rho \sin \vartheta, \rho \cos \vartheta, \rho z) \cos \vartheta \\
&\quad + 2\varepsilon \frac{1}{\rho} z_\varkappa F_0(\rho \cos \vartheta, \rho z^{(2)}) \cos \vartheta, \\
\frac{dz_{k+\varkappa}}{d\tau} &= -a_{\varkappa 1} z_1 - \ldots - a_{\varkappa k} z_k - \frac{1}{\rho} z_{k+\varkappa} U(\rho \sin \vartheta, \rho \cos \vartheta, \rho z) \cos \vartheta \\
&\quad + \frac{1}{\rho} V_\varkappa(\rho \sin \vartheta, \rho \cos \vartheta, \rho z) \\
&\quad + 2\varepsilon \frac{1}{\rho} z_{k+\varkappa} F_0(\rho \cos \vartheta, \rho z^{(2)}) \cos \vartheta - 2\varepsilon \frac{1}{\rho} F_\varkappa(\rho \cos \vartheta, \rho z^{(2)}) \\
&\qquad (\varkappa = 1, \ldots, k),
\end{aligned} \tag{1.5}$$

$$\rho^2 [1 + W(z) + \rho S(\rho, \vartheta, z)] = \mu^2. \tag{1.6}$$

Here $\mathbf{z}$ and $\mathbf{z}^{(2)}$ are vectors with the components $z_1, \ldots, z_{2k}$ and $z_{k+1}, \ldots, z_{2k}$, respectively, and $S = \rho^{-3}S_3$. We assume henceforth that the inequality

$$1 - \frac{1}{\rho}[U(\rho \sin\vartheta, \rho\cos\vartheta, \rho\mathbf{z}) - 2\varepsilon F_0(\rho\cos\vartheta, \rho\mathbf{z}^{(2)})]\sin\vartheta > 0 \quad (1.7)$$

holds in a parallelepiped $|\mathbf{z}| \leqslant b$, $0 \leqslant \rho \leqslant r_0$ and for any $\vartheta \geqslant \vartheta_0$ for all $\varepsilon \in [0, \varepsilon_0]$. Taking (1.7) into account, we divide the second and all the subsequent equations of system (1.5) by the first

$$\frac{d\rho}{d\vartheta} = \frac{U\cos\vartheta - 2\varepsilon F_0 \cos\vartheta}{1 - \rho^{-1}(U - 2\varepsilon F_0)\sin\vartheta},$$

$$\frac{dz_\varkappa}{d\vartheta} = \frac{z_{k+\varkappa} - \rho^{-1}z_\varkappa(U - 2\varepsilon F_0)\cos\vartheta}{1 - \rho^{-1}(U - 2\varepsilon F_0)\sin\vartheta},$$

$$\frac{dz_{k+\varkappa}}{d\vartheta} = [1 - \rho^{-1}(U - 2\varepsilon F_0)\sin\vartheta]^{-1}(-a_{\varkappa 1}z_1 - \ldots - a_{\varkappa k}z_k$$
$$-\rho^{-1}z_{k+\varkappa}U\cos\vartheta + \rho^{-1}V_\varkappa + 2\varepsilon\rho^{-1}z_{k+\varkappa}F_0\cos\vartheta - 2\varepsilon\rho^{-1}F_\varkappa) \quad (1.8)$$
$$(\varkappa = 1, \ldots, k).$$

The unperturbed system (1.8) (for $\varepsilon = 0$) was reduced in Chapter I, Section 1 to a quasilinear nonautonomous system of order $2k$ by using integral (I, 1, 1.6); it was solved in Chapter III for sufficiently small $\mu > 0$ in (1.6) by the methods of small parameter.

1.2. Complete system of variational equations in the Poincaré parameter and its solution. Let us represent system (1.8) as a vector equation

$$\frac{d\mathbf{x}}{d\vartheta} = \mathbf{f}(\vartheta, \mathbf{x}; \varepsilon), \quad (2.1)$$

where $\mathbf{x}$ is a vector with the components $\rho, z_1, \ldots, z_{2k}$; $\mathbf{f}$ is a vector-function formed from the right-hand sides of system (1.8), that is, analytic with respect to $\mathbf{x}$ and ε in the domain of definition of (1.7); and the coefficients of power series in $\rho, z_1, \ldots, z_{2k}$* are 2π-periodic functions of ϑ.

Let us assume that we know a solution $\mathbf{x}_0(\vartheta)$ of the unperturbed system (2.1) (i.e. for $\varepsilon = 0$)

$$\frac{d\mathbf{x}_0}{d\vartheta} = \mathbf{f}(\vartheta, \mathbf{x}_0; 0). \quad (2.2)$$

On the basis of the Poincaré theorem ([188a], vol. I, Ch. II), we seek a solution of system (2.1) for sufficiently small positive ε in the form

$$\mathbf{x} = \sum_{m=0}^{\infty} \varepsilon^m \mathbf{x}_m(\vartheta). \quad (2.3)$$

* We assume henceforth that $2k + 1 = n$ and n is any natural number.

Subtracting identity (2.2) from equation (2.1) and applying the Taylor formula for multivariable functions, we obtain

$$\sum_{m=1}^{\infty} \varepsilon^m \frac{d\mathbf{x}_m}{d\vartheta} = \sum_{\nu=1}^{\infty} \frac{1}{\nu!}\left(\frac{\partial}{\partial \mathbf{x}} \sum_{m=1}^{\infty} \varepsilon^m \mathbf{x}_m + \frac{\partial}{\partial \varepsilon}\varepsilon\right)^{\nu} \mathbf{f}(\vartheta,\ \mathbf{x}_0;\ 0).$$

Equating coefficients on terms containing identical powers of ε, we obtain a sequence of vector differential equations (*a complete system of variational equations in the Poincaré parameter*)

$$\frac{d\mathbf{x}_1}{d\vartheta} = \left(\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right)_0 \mathbf{x}_1 + \left(\frac{\partial \mathbf{f}}{\partial \varepsilon}\right)_0,$$

$$\frac{d\mathbf{x}_2}{d\vartheta} = \left(\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right)_0 \mathbf{x}_2 + \frac{1}{2}\left(\frac{\partial^2 \mathbf{f}}{\partial \mathbf{x}^2}\right)_0 \mathbf{x}_1\mathbf{x}_1 + \left(\frac{\partial^2 \mathbf{f}}{\partial \mathbf{x}\,\partial \varepsilon}\right)_0 \mathbf{x}_1 + \frac{1}{2}\left(\frac{\partial^2 \mathbf{f}}{\partial \varepsilon^2}\right)_0,$$

$$\begin{aligned}\frac{d\mathbf{x}_3}{d\vartheta} &= \left(\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right)_0 \mathbf{x}_3 + \frac{1}{2}\left(\frac{\partial^2 \mathbf{f}}{\partial \mathbf{x}^2}\right)_0 (\mathbf{x}_1\mathbf{x}_2 + \mathbf{x}_2\mathbf{x}_1) + \left(\frac{\partial^2 \mathbf{f}}{\partial \mathbf{x}\,\partial \varepsilon}\right)_0 \mathbf{x}_2 \\ &+ \frac{1}{6}\left(\frac{\partial^3 \mathbf{f}}{\partial \mathbf{x}^3}\right)_0 \mathbf{x}_1\mathbf{x}_1\mathbf{x}_1 + \frac{1}{2}\left(\frac{\partial^3 \mathbf{f}}{\partial \mathbf{x}^2\,\partial \varepsilon}\right)_0 \mathbf{x}_1\mathbf{x}_1 + \frac{1}{2}\left(\frac{\partial^3 \mathbf{f}}{\partial \mathbf{x}\,\partial \varepsilon^2}\right)_0 \mathbf{x}_1 + \frac{1}{6}\left(\frac{\partial^3 \mathbf{f}}{\partial \varepsilon^3}\right)_0,\end{aligned}$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$\begin{aligned}\frac{d\mathbf{x}_m}{d\vartheta} &= \left(\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right)_0 \mathbf{x}_m + \frac{1}{2}\left(\frac{\partial^2 \mathbf{f}}{\partial \mathbf{x}^2}\right)_0 \sum_{\alpha_1+\alpha_2=m} \mathbf{x}_{\alpha_1}\mathbf{x}_{\alpha_2} + \left(\frac{\partial^2 \mathbf{f}}{\partial \mathbf{x}\,\partial \varepsilon}\right)_0 \mathbf{x}_{m-1} \\ &+ \frac{1}{6}\left(\frac{\partial^3 \mathbf{f}}{\partial \mathbf{x}^3}\right)_0 \sum_{\alpha_1+\alpha_2+\alpha_3=m} \mathbf{x}_{\alpha_1}\mathbf{x}_{\alpha_2}\mathbf{x}_{\alpha_3} + \frac{1}{2}\left(\frac{\partial^3 \mathbf{f}}{\partial \mathbf{x}^2\,\partial \varepsilon}\right)_0 \sum_{\alpha_1+\alpha_2=m-1} \mathbf{x}_{\alpha_1}\mathbf{x}_{\alpha_2} \\ &+ \frac{1}{2}\left(\frac{\partial^3 \mathbf{f}}{\partial \mathbf{x}\,\partial \varepsilon^2}\right)_0 \mathbf{x}_{m-2} + \dots + \frac{1}{s!}\left(\frac{\partial^s \mathbf{f}}{\partial \mathbf{x}^s}\right)_0 \sum_{\alpha_1+\dots+\alpha_s=m} \mathbf{x}_{\alpha_1}\dots\mathbf{x}_{\alpha_s} \\ &+ \frac{1}{(s-1)!}\left(\frac{\partial^s \mathbf{f}}{\partial \mathbf{x}^{s-1}\,\partial \varepsilon}\right)_0 \sum_{\alpha_1+\dots+\alpha_{s-1}=m-1} \mathbf{x}_{\alpha_1}\dots\mathbf{x}_{\alpha_{s-1}} + \\ &\dots + \frac{1}{(s-l)!\,l!}\left(\frac{\partial^s \mathbf{f}}{\partial \mathbf{x}^{s-l}\,\partial \varepsilon^l}\right)_0 \sum_{\alpha_1+\dots+\alpha_{s-l}=m-l} \mathbf{x}_{\alpha_1}\dots\mathbf{x}_{\alpha_{s-l}} + \\ &\dots + \frac{1}{(s-1)!}\left(\frac{\partial^s \mathbf{f}}{\partial \mathbf{x}\,\partial \varepsilon^{s-1}}\right)_0 \mathbf{x}_{m-s+1} + \dots + \frac{1}{m!}\left(\frac{\partial^m \mathbf{f}}{\partial \mathbf{x}^m}\right)_0 \mathbf{x}_1^m \\ &+ \frac{1}{(m-1)!}\left(\frac{\partial^m \mathbf{f}}{\partial \mathbf{x}^{m-1}\,\partial \varepsilon}\right)_0 \mathbf{x}_1^{m-1} + \dots + \frac{1}{(m-l)!\,l!}\left(\frac{\partial^m \mathbf{f}}{\partial \mathbf{x}^{m-l}\,\partial \varepsilon^l}\right)_0 \mathbf{x}_1^{m-l} + \\ &\dots + \frac{1}{(m-1)!}\left(\frac{\partial^m \mathbf{f}}{\partial \mathbf{x}\,\partial \varepsilon^{m-1}}\right)_0 \mathbf{x}_1 + \frac{1}{m!}\left(\frac{\partial^m \mathbf{f}}{\partial \varepsilon^m}\right)_0. \qquad (2.4)\end{aligned}$$

The zero subscript indicates evaluation of partial derivatives for $\mathbf{x}_0 = \mathbf{x}_0(\vartheta)$ and $\varepsilon = 0$; $\alpha_1, \alpha_2, \dots$ are natural numbers; and the matrix

$$\frac{\partial \mathbf{f}}{\partial \mathbf{x}} = \left\| \frac{\partial f_j}{\partial \xi_h} \right\|_1^n, \quad \mathbf{f} = \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_n \end{pmatrix}.$$

The remaining terms in equations (2.4) must also be understood as operators; for example,

$$\frac{\partial^2 \mathbf{f}}{\partial \mathbf{x}^2}\mathbf{x}_1\mathbf{x}_2 = \frac{\partial\left[\left(\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right)\mathbf{x}_1\right]}{\partial \mathbf{x}}\mathbf{x}_2.$$

In general, these operators are noncommutative, so that, say, the terms with $\mathbf{x}_1\mathbf{x}_2$ and $\mathbf{x}_2\mathbf{x}_1$ are considered distinct.

If ε is a linear element of (2.1), that is, if

$$\mathbf{f}(\vartheta, \mathbf{x}; \varepsilon) = \mathbf{g}(\vartheta, \mathbf{x}) + \varepsilon \mathbf{h}(\vartheta, \mathbf{x}),$$

then equations (2.4) become

$$\frac{d\mathbf{x}_1}{d\vartheta} = \left(\frac{\partial \mathbf{g}}{\partial \mathbf{x}}\right)_0 \mathbf{x}_1 + \mathbf{h}(\vartheta, \mathbf{x}_0),$$

$$\frac{d\mathbf{x}_m}{d\vartheta} = \left(\frac{\partial \mathbf{g}}{\partial \mathbf{x}}\right)_0 \mathbf{x}_m + \sum_{\nu=2}^{m}\left[\frac{1}{\nu!}\left(\frac{\partial^\nu \mathbf{g}}{\partial \mathbf{x}^\nu}\right)_0 \sum_{\alpha_1+\ldots+\alpha_\nu=m} \mathbf{x}_{\alpha_1}\ldots\mathbf{x}_{\alpha_\nu}\right.$$

$$\left. + \frac{1}{(\nu-1)!}\left(\frac{\partial^{\nu-1}\mathbf{h}}{\partial \mathbf{x}^{\nu-1}}\right)_0 \sum_{\alpha_1+\ldots+\alpha_{\nu-1}=m-1} \mathbf{x}_{\alpha_1}\ldots\mathbf{x}_{\alpha_{\nu-1}}\right] \quad (m>1). \tag{2.5}$$

Successive integration of equations (2.4) and (2.5) is possible only in the scalar case.

If, however, the general integral of the unperturbed system (2.1) (i.e. for $\varepsilon = 0$) is known, the integration of system (2.4) or (2.5) of any order was shown by Poincaré to be reducible to quadratures. Indeed, let

$$\mathbf{x}_0 = \mathbf{x}_0(\vartheta; \mathbf{a}),$$

where $\mathbf{a}$ is an n-dimensional vector, be the general integral of equation (2.1) for $\varepsilon = 0$, that is,

$$\frac{d\mathbf{x}_0(\vartheta; \mathbf{a})}{d\vartheta} = \mathbf{f}(\vartheta, \mathbf{x}_0(\vartheta; \mathbf{a}); 0).$$

Differentiating this identity with respect to $\mathbf{a}$, we obtain

$$\frac{d\left(\frac{\partial \mathbf{x}_0}{\partial \mathbf{a}}\right)}{d\vartheta} = \left(\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right)_0 \frac{\partial \mathbf{x}_0}{\partial \mathbf{a}}.$$

Hence, $\partial\mathbf{x}_0/\partial\mathbf{a}$ is the fundamental matrix of each of the homogeneous systems of differential equations (2.4) (or (2.5)). The solution of the first system of (2.4) (and likewise of (2.5)) with zero initial conditions $\mathbf{x}_1(\vartheta_0) = \mathbf{0}$ (because $\mathbf{x}_0(\vartheta_0; \mathbf{a}) = \mathbf{x}^{(0)}$, $\mathbf{x}_\nu(\vartheta_0) = \mathbf{0}$, $\nu =$

= 1, 2, . . .) can then be written by the Lagrange method of variation of constants in the form

$$\mathbf{x}_1 = \frac{\partial \mathbf{x}_0}{\partial \mathbf{a}} \int_{\vartheta_0}^{\vartheta} \left(\frac{\partial \mathbf{x}_0}{\partial \mathbf{a}}\right)^{-1} \left(\frac{\partial \mathbf{f}}{\partial \varepsilon}\right)_0 d\vartheta, \tag{2.6}$$

where $\partial \mathbf{x}_0/\partial \mathbf{a}$ is a nonsingular matrix by virtue of the fact that the solution $\mathbf{x}_0 = \mathbf{x}_0(\vartheta;\ \mathbf{a})$ is general. In the case of $\mathbf{x}_m$ $(m > 1)$ the formulas obtained are similar to (2.6), where $(\partial \mathbf{f}/\partial \varepsilon)_0$ in the integrand is replaced by the inhomogeneous term of the corresponding system (2.4) or (2.5).

In order to find a solution of the initial system (1.1) we need to integrate the first equation of (1.5) with the same accuracy, within $O(\varepsilon^2)$, if we wish to determine the first corrections.

1.3. Vibrations in mechanical systems with one degree of freedom and different types of nonlinearity. We consider a mechanical system with one degree of freedom that is subject to restoring and resistance forces. Both of these functions are assumed to be analytic functions of the coordinates and velocity, respectively. We emphasize that only one of these forces, namely the resistance force, is assumed to be "ε-small", the upper bound for $\varepsilon > 0$ being determined by the majorant inequality of the Poincaré theorem ([188a], vol. I, Ch. II), which in general is ineffective. As for the restoring force, its nonlinear part determines the upper bound on the parameter μ, that is, the square root of the initial value of the reduced energy. The general solution suggested in the final paragraphs of Subsection 1.3 is thus valid for sufficiently small $\varepsilon > 0$ and for effectively restricted values of the initial energy of the system.

With a suitable time scale τ, the equation of motion of the system in question can be written as

$$\ddot{u} + 2\varepsilon F(\dot{u}) + u \quad U(u) = 0 \quad \left(\varepsilon > 0, \quad = \frac{d}{d\tau}\right), \tag{3.1}$$

where $F(\dot{u})$ and $U(u)$ are analytic functions in the whole range of the variables; their expansions begin with terms of power not lower than one and two, respectively; furthermore, the work of the resistance force is assumed negative over any displacement: $-F(\dot{u})\,\dot{u} < 0$, which holds, in particular, for the odd function $F(\dot{u})$. The unperturbed equation (3.1) possesses, for $\varepsilon = 0$, the energy integral

$$H = \dot{u}^2 + u^2 + 2V(u) = \mu^2 \quad \left(V(u) = -\int_0^u U(u)\, du \quad \mu > 0\right). \tag{3.2}$$

The Lyapunov substitution (1.4) transforms equation (3.1) to system (1.5) in the form

$$\frac{d\vartheta}{d\tau} = 1 - \frac{1}{\rho}\left[U(\rho\sin\vartheta) - 2\varepsilon F(\rho\cos\vartheta)\right]\sin\vartheta,$$

$$\frac{d\rho}{d\tau} = U(\rho\sin\vartheta)\cos\vartheta - 2\varepsilon F(\rho\cos\vartheta)\cos\vartheta. \tag{3.3}$$

We assume henceforth that for all ε and ρ in the rectangle $0 \leqslant \varepsilon \leqslant \varepsilon_0$ and $0 \leqslant \rho \leqslant r_0$ and for any ϑ the following inequality holds

$$1 - \frac{1}{\rho}\left[U(\rho\sin\vartheta) - 2\varepsilon F(\rho\cos\vartheta)\right]\sin\vartheta > 0 \tag{3.4}$$

Together with the first two formulas of (1.4), inequality (3.4) defines the allowed range of variation of u and $\dot{u}$. Under condition (3.4), we divide the second equation of (3.3) by the first

$$\frac{d\rho}{d\vartheta} = \frac{U(\rho\sin\vartheta)\cos\vartheta - 2\varepsilon F(\rho\cos\vartheta)\cos\vartheta}{1 - \frac{1}{\rho}\left[U(\rho\sin\vartheta) - 2\varepsilon F(\rho\cos\vartheta)\right]\sin\vartheta} \tag{3.5}$$

The integral of equation (3.5) for $\varepsilon = 0$ is found by substituting (1.4) into the energy integral (3.2); this yields

$$\rho^2 + 2V(\rho\sin\vartheta) = \mu^2; \tag{3.6}$$

note that the expansion of the second addend begins with terms of power not lower than three in ρ. Since $\rho \geqslant 0$, equation (3.6) has the unique solution

$$\rho = \rho_0(\vartheta;\mu) = \mu - \frac{1}{\mu} V(\mu\sin\vartheta) + \frac{1}{\mu^2}\left(\frac{\partial V}{\partial\rho}\right)_{\rho=\mu} V(\mu\sin\vartheta)\sin\vartheta - \frac{2}{\mu^3}\left[V(\mu\sin\vartheta)\right]^2 + O(\mu^4) \tag{3.7}$$

for sufficiently small positive μ (the radius of convergence of series (3.7) will be found later in the example discussed in Subsection 1.4).

On the basis of the Poincaré theorem ([188a], vol. I, Ch. II), we seek a solution of (3.5) for sufficiently small positive ε in the form

$$\rho = \sum_{m=0}^{\infty} \rho_m(\vartheta;\mu)\,\varepsilon^m. \tag{3.8}$$

A sequence of the differential equations (2.4) for determining $\rho_m(\vartheta;\mu)$ $(m = 1, 2, \ldots)$ (a complete set of variational equations in the Poincaré parameter) was derived in Subsection 1.2. We are now interested in the first correction $\rho_1(\vartheta;\mu)$; an appropriate differential equation for this correction is a variational equation in the

Poincaré parameter (the first equation of (2.4)). Let us derive it directly. Recasting (3.5) into

$$\rho' = f(\vartheta, \rho; \varepsilon) \quad \left({}' = \frac{d}{d\vartheta}\right)$$

and subtracting from it the identity

$$\rho_0' = f(\vartheta, \rho_0; 0),$$

we obtain

$$r' = \left(\frac{\partial f}{\partial \rho}\right)_0 r + \left(\frac{\partial f}{\partial \varepsilon}\right)_0 + O_1(\varepsilon^2) \quad (r \equiv \rho(\vartheta; \mu) - \rho_0(\vartheta; \mu)), \tag{3.9}$$

where the zero subscript indicates that the partial derivatives are evaluated at $\rho = \rho_0(\vartheta; \mu)$ and $\varepsilon = 0$. Dividing (3.9) by ε and then setting $\varepsilon = 0$, we arrive, by virtue of (3.8), at a variational equation in the Poincaré parameter

$$\rho_1' = \left(\frac{\partial f}{\partial \rho}\right)_0 \rho_1 + \left(\frac{\partial f}{\partial \varepsilon}\right)_0. \tag{3.10}$$

In the scalar case we are discussing now, this equation is readily integrated, although the resultant form is inconvenient to apply. Therefore it is preferable to resort to the Poincaré method ([188a], vol. I, Ch. II) presented at the end of Subsection 1.2. In our case equation (3.7) is the general integral of the unperturbed-motion equation (3.5) (i.e. for $\varepsilon = 0$)

$$\rho_0' = f(\vartheta, \rho_0(\vartheta; \mu); 0).$$

Differentiating this identity with respect to μ, we obtain

$$\frac{d\left(\frac{\partial \rho_0}{\partial \mu}\right)}{d\vartheta} = \left(\frac{\partial f}{\partial \rho}\right)_0 \frac{\partial \rho_0}{\partial \mu},$$

that is, $\partial\rho_0/\partial\mu$ is a solution of a homogeneous equation corresponding to (3.10). Then a solution of (3.10) with the zero initial conditions $\rho_1(\vartheta_0; \mu) = 0$ (since $\rho_0(\vartheta_0; \mu) = \rho^{(0)}$; $\rho_m(\vartheta_0; \mu) = 0, m = 1, 2, \ldots$) can be written in the form of (2.6)

$$\rho_1 = \frac{\partial \rho_0}{\partial \mu} \int_{\vartheta_0}^{\vartheta} \left(\frac{\partial \rho_0}{\partial \mu}\right)^{-1} \left(\frac{\partial f}{\partial \varepsilon}\right)_0 d\vartheta. \tag{3.11}$$

Expressions (3.7) and (3.5) yield

$$\frac{\partial \rho_0}{\partial \mu} = 1 - \frac{\partial}{\partial \mu}\left[\frac{1}{\mu} V(\mu \sin \vartheta)\right] + O(\mu^2), \tag{3.12}$$

$$\left(\frac{\partial f}{\partial \varepsilon}\right)_0 = -2\left[1 - \frac{1}{\rho_0} U(\rho_0 \sin \vartheta) \sin \vartheta\right]^{-1} F(\rho_0 \cos \vartheta)$$
$$\times \cos \vartheta \left\{1 + \frac{1}{\rho_0} U(\rho_0 \sin \vartheta) \sin \vartheta \left[1 - \frac{1}{\rho_0} U(\rho_0 \sin \vartheta) \sin \vartheta\right]^{-1}\right\}. \tag{3.13}$$

Now we derive from expressions (1.4), (3.7), (3.8), and (3.11)

$$u = [\rho_0(\vartheta;\ \mu) + \varepsilon \rho_1(\vartheta;\ \mu) + O(\varepsilon^2)] \sin \vartheta. \tag{3.14}$$

After determining $\vartheta = \vartheta(\tau;\ \vartheta_0)$ from the first equation of (3.3), we find the general solution of the initial equation (3.1) (with the arbitrary constants μ and ϑ_0) for sufficiently small positive ε determined by condition (3.4) and the Poincaré theorem cited above. The upper bound for $\mu > 0$ (square root from the initial value of the reduced energy (3.2)) is determined by the radius of convergence of series (3.7).

1.4. The Duffing equation with linear damping. As an example, we consider the equation

$$\ddot{u} + 2\varepsilon \dot{u} + u + \alpha u^3 = 0 \quad (\varepsilon > 0,\ \alpha > 0), \tag{4.1}$$

that is (see equations (3.1) and (3.2)), $F(\dot{u}) = \dot{u}$, $U(u) = -\alpha u^3$, and $V(u) = \frac{1}{4}\alpha u^4$. Condition (3.4) defines the range of allowed values of variables $\rho \geqslant 0$, $\vartheta > \vartheta_0$, and parameter $\varepsilon \geqslant 0$ in a three-dimensional octant

$$1 + \alpha \rho^2 \sin^4 \vartheta + \varepsilon \sin 2\vartheta > 0 \tag{4.2}$$

and is always satisfied for $0 \leqslant \varepsilon \leqslant 1$. The energy integral (3.2) of the unperturbed equation (4.1) ($\varepsilon = 0$) becomes

$$\rho^2 + \frac{1}{2}\alpha \rho^4 \sin^4 \vartheta = \mu^2$$

so that under the condition $2\alpha\mu^2 \sin^4 \vartheta < 1$, which is always satisfied if

$$\mu^2 < \frac{1}{2\alpha}, \tag{4.3}$$

this integral has the unique solution (3.7)

$$\rho_0(\vartheta;\ \mu) = \mu\left[1 - \frac{1}{4}\alpha\mu^2 \sin^4 \vartheta + O(\mu^4)\right]. \tag{4.4}$$

We now compute (3.12) and (3.13)

$$\frac{\partial \rho_0}{\partial \mu} = 1 - \frac{3}{4}\alpha\mu^2 \sin^4 \vartheta + O(\mu^4),$$

$$\left(\frac{\partial f}{\partial \varepsilon}\right)_0$$

$$= -2\rho_0 \cos^2 \vartheta (1 + \alpha\rho_0^2 \sin^4 \vartheta)^{-1} [1 - \alpha\rho_0^2 \sin^4 \vartheta (1 + \alpha\rho_0^2 \sin^4 \vartheta)^{-1}]$$

$$= -2\mu \cos^2 \vartheta \left[1 - \frac{9}{4}\alpha\mu^2 \sin^4 \vartheta + O(\mu^4)\right].$$

We wish to find a solution that satisfies the initial conditions $u(0) = 0$, and $\dot{u}(0) = \dot{u}_0 > 0$; this entails, by virtue of (1.4) and (4.4), $\vartheta_0 = 0$ and $\mu = \dot{u}_0$. We also assume (see (4.3)) that $\dot{u}_0 < \sqrt{0.5\alpha^{-1}}$. Formula (3.11) gives

$$\rho_1(\vartheta; \mu) = -\mu\left[\vartheta + \frac{1}{2}\sin 2\vartheta + \frac{1}{4}\alpha\mu^2\left(-3\vartheta \sin^4 \vartheta \right.\right.$$

$$\left.\left. - \frac{3}{2}\sin 2\vartheta \sin^4 \vartheta - \frac{3}{4}\vartheta + \frac{3}{16}\sin 4\vartheta + \frac{1}{4}\sin^3 2\vartheta\right) + O(\mu^4)\right].$$

The first equation of (3.3) yields

$$\tau = \int_0^{\vartheta} (1 - \alpha\mu^2 \sin^4 \vartheta - \varepsilon \sin 2\vartheta + \ldots)\, d\vartheta$$

$$= \vartheta - \varepsilon \sin^2 \vartheta - \alpha\mu^2\left(\frac{1}{32}\sin 4\vartheta - \frac{1}{4}\sin 2\vartheta + \frac{3}{8}\vartheta\right) + \ldots,$$

where the terms of power ε^2 and $\varepsilon\mu^2$ are ignored. Inverting the last formula, we obtain

$$\vartheta = \tau + \frac{1}{4}\alpha\mu^2\left(\frac{3}{2}\tau - \sin 2\tau + \frac{1}{8}\sin 4\tau\right) + \varepsilon \sin^2 \tau + \ldots .$$

Let us rewrite the solution in the form of (3.14)

$$u = \left[\mu\left(1 - \frac{1}{4}\alpha\mu^2 \sin^4 \tau\right) - \varepsilon\mu\left(\tau + \frac{1}{2}\sin 2\tau\right) + O(\varepsilon^2)\right]$$

$$\times \sin\left[\tau + \frac{1}{4}\alpha\mu^2\left(\frac{3}{2}\tau - \sin 2\tau + \frac{1}{8}\sin 4\tau\right) + \varepsilon \sin^2 \tau + O(\varepsilon^2)\right], \quad (4.5)$$

where the range of τ is assumed to be of the order of $O(\varepsilon^{-1})$.

Remark. In the specific case analyzed above, namely that of linear damping, we can obtain a solution valid for any $\tau > 0$. The substitution

$$u = \exp(-\varepsilon\tau)\, v$$

transforms (4.1) to the form

$$\frac{d\mathbf{x}}{d\tau} = \mathbf{g}(\tau, \mathbf{x}; \alpha),$$

$$\mathbf{x} = \begin{pmatrix} v \\ \dot{v} \end{pmatrix}, \quad \mathbf{g} = \begin{pmatrix} \dot{v} \\ -(1-\varepsilon^2)\, v - \alpha e^{-2\varepsilon\tau} v^3 \end{pmatrix}. \tag{4.6}$$

We want the general solution of (4.6) in the form

$$\mathbf{x} = \mathbf{x}_0(\tau) + \alpha \mathbf{x}_1(\tau) + \alpha^2 \mathbf{x}_2(\tau) + \ldots$$

The general solution of the unperturbed equation (4.6) (i.e. for $\alpha = 0$) for $0 < \varepsilon < 1$ is

$$\mathbf{x}_0(\tau) = \begin{pmatrix} M\cos R\tau + N\sin R\tau \\ -MR\sin R\tau + NR\cos R\tau \end{pmatrix} \quad (R \equiv \sqrt{1-\varepsilon^2}).$$

The first correction is obtained from formula (2.8)

$$\mathbf{x}_1 = \frac{\partial \mathbf{x}_0}{\partial \mathbf{a}} \int_0^\tau \left(\frac{\partial \mathbf{x}_0}{\partial \mathbf{a}}\right)^{-1} \left(\frac{\partial \mathbf{g}}{\partial \alpha}\right)_0 d\tau, \tag{4.7}$$

where **a** is a vector with the components M and N, and

$$\frac{\partial \mathbf{x}_0}{\partial \mathbf{a}} = \begin{Vmatrix} \dfrac{\partial v_0}{\partial M} & \dfrac{\partial v_0}{\partial N} \\[2ex] \dfrac{\partial \dot{v}_0}{\partial M} & \dfrac{\partial \dot{v}_0}{\partial N} \end{Vmatrix}.$$

The zero subscript indicates evaluation at $\alpha = 0$, $\mathbf{x} = \mathbf{x}_0(\tau)$. The general solution of equation (4.1) for $0 < \varepsilon < 1$ is

$$u = [M\cos R\tau + N\sin R\tau + \alpha v_1(\tau) + O(\alpha^2)]\, e^{-\varepsilon\tau}, \tag{4.8}$$

where

$$\begin{aligned} v_1(\tau) = \frac{1}{16R} \{ & 2\,[e^{-2\varepsilon\tau}(-\varepsilon\sin 2R\tau - R\cos 2R\tau) + R] \\ & \times [M(M^2 + 3N^2)\cos R\tau - N(N^2 + 3M^2)\sin R\tau] \\ & + (4 - 3\varepsilon^2)^{-1}\,[e^{-2\varepsilon\tau}(-\varepsilon\sin 4R\tau - 2R\cos 4R\tau) + 2R] \\ & \times [M(M^2 - 3N^2)\cos R\tau + N(N^2 - 3M^2)\sin R\tau] \\ & + (4 - 3\varepsilon^2)^{-1}\,[e^{-2\varepsilon\tau}(-\varepsilon\cos 4R\tau + 2R\sin 4R\tau) + \varepsilon] \\ & \times [N(N^2 - 3M^2)\cos R\tau - M(M^2 - 3N^2)\sin R\tau] \\ & - 4\,[e^{-2\varepsilon\tau}(-\varepsilon\cos 2R\tau + R\sin 2R\tau) + \varepsilon] \\ & \times (N^3\cos R\tau + M^3\sin R\tau) + 3(M^2 + N^2)(1 - e^{-2\varepsilon\tau}) \\ & \times (N\cos R\tau - M\sin R\tau)\} \end{aligned}$$

(obvious computations in formula (4.7) are omitted). The solution for $\varepsilon > 1$ is readily obtained. Obviously, $M = u(0)$ and $N = (1 - \varepsilon^2)^{-1/2} \dot{u}(0)$.

1.5. Spring-loaded pendulum with linear damping. We consider a plane spring-loaded pendulum of mass m suspended from a weightless spring of unstressed length l and with force constant c (Fig. 9). Hook's law is assumed valid. Let x and $y' = l + \lambda + y$ be the Cartesian coordinates of the point mass m relative to the suspension point O; where $\lambda = mg/c$ is the static elongation of the spring. The additive constant of the potential energy Π of the force of gravity and the elastic force of the spring is chosen so that it vanishes at the static equilibrium point $x = y = 0$; consequently,

$$\Pi = -mgy + \frac{1}{2} c \left[\sqrt{x^2 + (l + \lambda + y)^2} - l\right]^2 - \frac{1}{2} c\lambda^2,$$

$$K = \frac{1}{2} m \left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right].$$

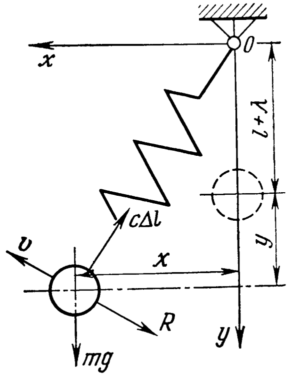

FIG. 9

We assume that a resistance force R proportional to velocity is also applied to the point mass: $R = -bv$. We denote the cyclic frequency of vertical vibrations of the spring-suspended mass by $\omega = \sqrt{c/m}$ and introduce dimensionless time $\tau = \omega t$ and coordinates $\xi = x/l$ and $\eta = y/l$. The equations of motion then become

$$\frac{d^2\eta}{d\tau^2} + \eta - \left[\frac{1+\gamma+\eta}{\sqrt{\xi^2 + (1+\gamma+\eta)^2}} - 1\right] = -2\varepsilon \frac{d\eta}{d\tau},$$

$$\frac{d^2\xi}{d\tau^2} + \frac{\gamma}{1+\gamma}\xi - \left[\frac{1}{\sqrt{\xi^2 + (1+\gamma+\eta)^2}} - \frac{1}{1+\gamma}\right]\xi = -2\varepsilon \frac{d\xi}{d\tau}, \quad (5.1)$$

where

$$\gamma = \frac{\lambda}{l}, \quad \varepsilon = +\frac{b}{2\sqrt{cm}}$$

are dimensionless parameters and the expansions of the expressions in brackets in the neighbourhood of $\xi = \eta = 0$ begin with terms of power two. For $\varepsilon = 0$ the energy integral is

$$\frac{1}{cl}\sqrt{2(K + \Pi)c} = \mu = \text{const}. \quad (5.2)$$

The Lyapunov substitution

$$\eta=\rho\sin\vartheta,\quad \frac{d\eta}{d\tau}=\rho\cos\vartheta,\quad \xi=\rho z_1,\quad \frac{d\xi}{d\tau}=\rho z_2 \tag{5.3}$$

transforms the perturbed system (5.1) and integral (5.2) of the unperturbed system to the form of (1.8) and (1.6)

$$\frac{d\rho}{d\vartheta}=\frac{U\cos\vartheta-2\varepsilon\rho\cos^2\vartheta}{1-\rho^{-1}U\sin\vartheta+\varepsilon\sin 2\vartheta},$$

$$\frac{dz_1}{d\vartheta}=\frac{z_2-z_1\rho^{-1}U\cos\vartheta+2\varepsilon z_1\cos^2\vartheta}{1-\rho^{-1}U\sin\vartheta+\varepsilon\sin 2\vartheta},$$

$$\frac{dz_2}{d\vartheta}=\frac{-\gamma(1+\gamma)^{-1}z_1-z_2\rho^{-1}U\cos\vartheta+\rho^{-1}V-2\varepsilon z_2\sin^2\vartheta}{1-\rho^{-1}U\sin\vartheta+\varepsilon\sin 2\vartheta}, \tag{5.4}$$

$$\rho^2\left[1+\frac{\gamma}{1+\gamma}z_1^2+z_2^2+\rho S(\vartheta,\rho,z_1,z_2)\right]=\mu^2, \tag{5.5}$$

where

$$U=-\frac{\rho^2 z_1^2}{2(1+\gamma)^2}+O(\rho^3),\quad V=-\frac{\rho^2 z_1\sin\vartheta}{(1+\gamma)^2}+O(\rho^3)$$

and condition (1.7) is assumed satisfied, that is,

$$1+\varepsilon\sin 2\vartheta+\frac{1}{2}(1+\gamma)^{-2}\rho z_1^2\sin\vartheta+O(\rho^2)>0. \tag{5.6}$$

The unperturbed system (5.4) (i.e. for $\varepsilon=0$) has a generating solution (see III, 1, 2.3)

$$\rho_0(\vartheta;\ \mu,\ M,\ N)=\mu\,[1+g^2(M^2+N^2)]^{-1/2}+O(\mu^2),$$

$$z_1^0(\vartheta;\ \mu,\ M,\ N)=M\cos g\vartheta+N\sin g\vartheta+O(\mu),$$

$$z_2^0(\vartheta;\ \mu,\ M,\ N)=g(-M\sin g\vartheta+N\cos g\vartheta)+O(\mu), \tag{5.7}$$

where

$$g=+\sqrt{\frac{\gamma}{1+\gamma}}\quad\left(g\neq\frac{1}{2}\right).$$

As was demonstrated in Chapter III, Subsection 1.2, this solution is general for all γ with the exception of $\gamma=\frac{1}{3}$ $\left(g=\frac{1}{2}\right)$. In the example above, the vectors $\mathbf{a}$ and $\mathbf{x}_0$ are

$$\mathbf{a}=\begin{pmatrix}\mu\\ M\\ N\end{pmatrix},\quad \mathbf{x}_0=\begin{pmatrix}\rho_0\\ z_1^0\\ z_2^0\end{pmatrix}.$$

The elements of (2.6) are found to be

$$\frac{\partial \mathbf{x}_0}{\partial \mathbf{a}} = \left\| \begin{matrix} H^{-1/2}+O(\mu) & -g^2MH^{-3/2}\mu+O(\mu^2) & -g^2NH^{-3/2}\mu+O(\mu^2) \\ \varphi(\vartheta)+O(\mu) & \cos g\vartheta+O(\mu) & \sin g\vartheta+O(\mu) \\ \psi(\vartheta)+O(\mu) & -g\sin g\vartheta+O(\mu) & g\cos g\vartheta+O(\mu) \end{matrix} \right\|,$$

where

$$H = 1 + g^2(M^2+N^2).$$

In calculating the inverse matrix, we take a somewhat rough approximation, assuming $\varphi(\vartheta) = \psi(\vartheta) \equiv 0$; we then obtain

$$\left(\frac{\partial \mathbf{x}_0}{\partial \mathbf{a}}\right)^{-1} = \left\| \begin{matrix} H^{1/2}+O(\mu) & g^2H^{-1}L(\vartheta)\mu+O(\mu^2) & H^{-1}L'(\vartheta)\mu+O(\mu^2) \\ O(\mu) & \cos g\vartheta+O(\mu) & -g^{-1}\sin g\vartheta+O(\mu) \\ O(\mu) & \sin g\vartheta+O(\mu) & g^{-1}\cos g\vartheta+O(\mu) \end{matrix} \right\|,$$

where $L(\vartheta) \equiv M\cos g\vartheta + N\sin g\vartheta$.

The vector $(\partial \mathbf{f}/\partial\varepsilon)_0$, where $\mathbf{f}$ is a vector of the right-hand sides of system (5.4) and the zero subscript indicates evaluation at $\varepsilon = 0$ and the generating solution (5.7), is equal to

$$\left(\frac{\partial \mathbf{f}}{\partial \varepsilon}\right)_0 = \begin{pmatrix} -2\mu H^{-1/2}\cos^2\vartheta+O(\mu^2) \\ 2L(\vartheta)\cos^2\vartheta-L'(\vartheta)\sin 2\vartheta+O(\mu) \\ -2L'(\vartheta)\sin^2\vartheta+g^2L(\vartheta)\sin 2\vartheta+O(\mu) \end{pmatrix}.$$

In formula (2.6) we set $\vartheta_0 = 0$; according to (5.3), this means that at the initial moment $\tau = 0$ we have $\eta(0) = 0$. Omitting obvious computations in (2.6), we obtain the first two necessary components of the vector $\mathbf{x}_1(\vartheta; \mu, M, N)$ of the first correction

$$\rho_1(\vartheta; \mu, M, N) = -\mu H^{-1/2}\left(\vartheta+\frac{1}{2}\sin 2\vartheta\right)+O(\mu^2),$$

$$z_1^1(\vartheta; \mu, M, N) = \frac{M}{g}\sin g\vartheta+\frac{1}{2}M\sin 2\vartheta\cos g\vartheta+gM\sin^2\vartheta\sin g\vartheta$$
$$-\frac{1}{2}N\sin 2\vartheta\sin g\vartheta-gN\sin^2\vartheta\cos g\vartheta+O(\mu).$$

The integration of the first equation of (1.5), again for $\vartheta_0 = 0$, gives

$$\vartheta = \vartheta(\tau) = \tau+\varepsilon\sin^2\tau+O(\mu)+\ldots,$$

where the second-power terms are ignored in μ and ε. Finally, the solution of system (5.1) in the case $\eta(0) = 0$ is obtained in the form

$$\eta(\tau) = [\rho_0(\vartheta(\tau); \mu, M, N) + \varepsilon\rho_1(\vartheta(\tau); \mu, M, N)] \sin\vartheta(\tau) + O(\varepsilon^2),$$
$$\xi(\tau) = [\rho_0(\vartheta(\tau); \mu, M, N) + \varepsilon\rho_1(\vartheta(\tau); \mu, M, N)] \times [z_1^0(\vartheta(\tau); \mu, M, N) + \varepsilon z_1^1(\vartheta(\tau); \mu, M, N)] + O(\varepsilon^2).$$

The constants μ (the initial value of energy (5.2)), M, and N are determined from the initial conditions $\dot{\eta}(0)$, $\xi(0)$, and $\dot{\xi}(0)$; the range of τ is of the order of $O(\varepsilon^{-1})$.

§ 2. On Lyapunov-Type Systems

Definition 1. *A nearly Lyapunov system* is a real system of the type

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x} + \mathbf{X}(\mathbf{x}) + \mu\mathbf{F}(t, \mathbf{x}, \mu) \tag{0.1}$$

that reduces to a Lyapunov system (I, 1, 1.1) for $\mu = 0$, that is,

$$\mathbf{A} = \lambda\mathbf{J}_2 + \mathbf{P}, \quad \mathbf{J}_2 = \begin{Vmatrix} 0 & -1 \\ 1 & 0 \end{Vmatrix}, \quad \mathbf{P} = \| p_{sr} \|_1^n$$

and in which the vector-function $\mathbf{F}$ is continuous, analytic in $\mathbf{x}$ and in the small parameter μ in a given domain, and 2π-periodic in t.

Definition 2. *A Lyapunov-type system* is a real system of the type

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x} + \mathbf{X}(\mathbf{x}), \tag{0.2}$$

in which the matrix $\mathbf{A}$ of the linear part of the system has l zero eigenvalues with simple elementary divisors, two pure imaginary eigenvalues $\pm\lambda i$, and no eigenvalues that are multiples of $\pm\lambda i$; and $\mathbf{X}(\mathbf{x})$ is (as in (0.1)) an analytic vector-function of $\mathbf{x}$ whose expansion begins with terms of power not lower than two.

Lyapunov-type systems are shown in this section to be reducible to Lyapunov systems.

2.1. Statement of the problem. We present a Lyapunov-type system in detailed form, using notation distinct from that of (0.1)

$$\frac{d\mathbf{x}}{dt} = \mathbf{u}(\mathbf{x}, \mathbf{y}, \mathbf{z}), \quad \frac{d\mathbf{y}}{dt} = \lambda\mathbf{J}_2\mathbf{y} + \mathbf{v}(\mathbf{x}, \mathbf{y}, \mathbf{z}), \quad \frac{d\mathbf{z}}{dt} = \mathbf{B}\mathbf{z} + \mathbf{w}(\mathbf{x}, \mathbf{y}, \mathbf{z}), \tag{1.1}$$

where

$$\mathbf{x}=\begin{pmatrix}\xi_1\\ \cdot\\ \cdot\\ \cdot\\ \xi_l\end{pmatrix},\qquad \mathbf{y}=\begin{pmatrix}\eta_1\\ \eta_2\end{pmatrix},\qquad \mathbf{z}=\begin{pmatrix}\zeta_1\\ \cdot\\ \cdot\\ \cdot\\ \zeta_m\end{pmatrix},$$

$$\mathbf{u}=\begin{pmatrix}u_1\\ \cdot\\ \cdot\\ \cdot\\ u_l\end{pmatrix},\qquad \mathbf{v}=\begin{pmatrix}v_1\\ v_2\end{pmatrix},\qquad \mathbf{w}=\begin{pmatrix}w_1\\ \cdot\\ \cdot\\ \cdot\\ w_m\end{pmatrix},$$

$$\mathbf{J}_2=\begin{Vmatrix}0 & -1\\ 1 & 1\end{Vmatrix},\qquad \mathbf{B}=\| b_{jh}\|_1^m,$$

and the vector-functions $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are analytic functions of the variables $\xi_1, \ldots, \xi_l$, η_1, η_2, and $\zeta_1, \ldots, \zeta_m$ in some neighbourhood of the origin; power expansions of $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in these variables begin with terms of power not lower than two; the set of roots of equation

$$\det\,[\mathbf{B}-\varkappa\mathbf{I}_m]=0$$

includes neither zero elements nor multiples of $\pm\lambda i$.

We assume that in addition to the Lyapunov scalar integral

$$(\mathbf{y},\ \mathbf{y})+\psi\,(\mathbf{x},\ \mathbf{y},\ \mathbf{z})=\gamma_0 \tag{1.2}$$

system (1.1) also has l analytic first integrals given in vector form by

$$\mathbf{p}\,(\mathbf{x})+\mathbf{f}\,(\mathbf{x},\ \mathbf{y},\ \mathbf{z})=\mathbf{c}, \tag{1.3}$$

where

$$\mathbf{p}=\begin{pmatrix}\pi_1\\ \cdot\\ \cdot\\ \cdot\\ \pi_l\end{pmatrix},\qquad \mathbf{f}=\begin{pmatrix}\varphi_1\\ \cdot\\ \cdot\\ \cdot\\ \varphi_l\end{pmatrix},\qquad \mathbf{c}=\begin{pmatrix}\gamma_1\\ \cdot\\ \cdot\\ \cdot\\ \gamma_l\end{pmatrix}.$$

Expansions of ψ, $\varphi_1, \ldots, \varphi_l$ in (1.2) and (1.3) begin with terms of at least the second power with respect to $\xi_1, \ldots, \xi_l$, η_1, η_2, and $\zeta_1, \ldots, \zeta_m$; $\pi_1(x), \ldots, \pi_l(x)$ are linear independent forms of variables $\xi_1, \ldots, \xi_l$ that can be chosen in the form $\pi_j(x)\equiv\xi_j$ $(j=1, \ldots, l)$; consequently,

$$\mathbf{p}\,(\mathbf{x})\equiv\mathbf{x}. \tag{1.4}$$

2.2. Transformation of Lyapunov-type systems.

Lemma I. *System* (1.1)-(1.3) *can be transformed in such a way that the vector-functions* $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$, and $\mathbf{f}$ vanish for $\mathbf{y} = \mathbf{z} = 0$

$$\mathbf{u}(\mathbf{x}, 0, 0) = \mathbf{v}(\mathbf{x}, 0, 0) = \mathbf{w}(\mathbf{x}, 0, 0) = \mathbf{f}(\mathbf{x}, 0, 0) \equiv 0.$$

The proof of the lemma is based on the Lyapunov transformation ([108a], Sec. 28) in the critical case of a single zero root. Namely, we perform a substitution of variables in (1.1)-(1.3)

$$\mathbf{y}(t) = \tilde{\mathbf{y}}(t) + \mathbf{y}(\mathbf{x}), \quad \mathbf{z}(t) = \tilde{\mathbf{z}}(t) + \mathbf{z}(\mathbf{x}), \tag{2.1}$$

where $\mathbf{y}(\mathbf{x})$ and $\mathbf{z}(\mathbf{x})$ are analytic vector-functions of the variables $\xi_1, \ldots, \xi_l$; the vector-functions contain no linear terms and satisfy the equations

$$\lambda \mathbf{J}_2 \mathbf{y}(\mathbf{x}) + \mathbf{v}(\mathbf{x}, \mathbf{y}(\mathbf{x}), \mathbf{z}(\mathbf{x})) = 0,$$
$$\mathbf{B}\mathbf{z}(\mathbf{x}) + \mathbf{w}(\mathbf{x}, \mathbf{y}(\mathbf{x}), \mathbf{z}(\mathbf{x})) = 0.$$

Substitution of the new variables yields (new nonlinear terms are denoted by $\tilde{\mathbf{u}}$, $\tilde{\mathbf{v}}$, and $\tilde{\mathbf{w}}$)

$$\frac{d\mathbf{x}}{dt} = \mathbf{u}(\mathbf{x}, \tilde{\mathbf{y}} + \mathbf{y}(\mathbf{x}), \tilde{\mathbf{z}} + \mathbf{z}(\mathbf{x})) \equiv \tilde{\mathbf{u}}(\mathbf{x}, \tilde{\mathbf{y}}, \tilde{\mathbf{z}}),$$

$$\frac{d\tilde{\mathbf{y}}}{dt} = \lambda \mathbf{J}_2 \tilde{\mathbf{y}} + \mathbf{v}(\mathbf{x}, \tilde{\mathbf{y}} + \mathbf{y}(\mathbf{x}), \tilde{\mathbf{z}} + \mathbf{z}(\mathbf{x})) - \mathbf{v}(\mathbf{x}, \mathbf{y}(\mathbf{x}), \mathbf{z}(\mathbf{x})) - \frac{\partial \mathbf{y}}{\partial \mathbf{x}} \tilde{\mathbf{u}} \equiv \lambda \mathbf{J}_2 \tilde{\mathbf{y}} + \tilde{\mathbf{v}}(\mathbf{x}, \tilde{\mathbf{y}}, \tilde{\mathbf{z}}),$$

$$\frac{d\tilde{\mathbf{z}}}{dt} = \mathbf{B}\tilde{\mathbf{z}} + \mathbf{w}(\mathbf{x}, \tilde{\mathbf{y}} + \mathbf{y}(\mathbf{x}), \tilde{\mathbf{z}} + \mathbf{z}(\mathbf{x})) - \mathbf{w}(\mathbf{x}, \mathbf{y}(\mathbf{x}), \mathbf{z}(\mathbf{x})) - \frac{\partial \mathbf{z}}{\partial \mathbf{x}} \tilde{\mathbf{u}} \equiv \mathbf{B}\tilde{\mathbf{z}} + \tilde{\mathbf{w}}(\mathbf{x}, \tilde{\mathbf{y}}, \tilde{\mathbf{z}}), \tag{2.2}$$

where

$$\frac{\partial \mathbf{y}}{\partial \mathbf{x}} = \left\| \frac{\partial \eta_j}{\partial \xi_h} \right\|, \quad \frac{\partial \mathbf{z}}{\partial \mathbf{x}} = \left\| \frac{\partial \zeta_k}{\partial \xi_h} \right\|$$

are matrices of order $2 \times l$ and $m \times l$, respectively. We wish to prove that

$$\tilde{\mathbf{u}}(\mathbf{x}, 0, 0) = \tilde{\mathbf{v}}(\mathbf{x}, 0, 0) = \tilde{\mathbf{w}}(\mathbf{x}, 0, 0) \equiv 0.$$

By (1.4) and (2.1), the vector integral (1.3) can be transformed into

$$\mathbf{x} + \mathbf{f}(\mathbf{x}, \tilde{\mathbf{y}}(t) + \mathbf{y}(\mathbf{x}), \tilde{\mathbf{z}}(t) + \mathbf{z}(\mathbf{x})) = \mathbf{c}. \tag{2.3}$$

Differentiation of (2.3), by virtue of (2.2), yields

$$\tilde{u}(x, \tilde{y}, \tilde{z}) + \frac{\partial f}{\partial x}\tilde{u}(x, \tilde{y}, \tilde{z}) + \frac{\partial f}{\partial y}\left[\lambda J_2\tilde{y} + \tilde{v}(x, \tilde{y}, \tilde{z}) + \frac{\partial y}{\partial x}\tilde{u}(x, \tilde{y}, \tilde{z})\right]$$
$$+ \frac{\partial f}{\partial z}\left[B\tilde{z} + \tilde{w}(x, \tilde{y}, \tilde{z}) + \frac{\partial z}{\partial x}\tilde{u}(x, \tilde{y}, \tilde{z})\right] = 0.$$

We assume in these identities that $\tilde{y} = \tilde{z} = 0$ (note that it follows from (2.2) that $\tilde{v}(x, 0, 0) = -\frac{\partial y}{\partial x}\tilde{u}(x, 0, 0)$ and $\tilde{w}(x, 0, 0) = -\frac{\partial z}{\partial x}\tilde{u}(x, 0, 0)$)

$$\left[I_l + \frac{\partial f(x, y, z)}{\partial x}\right]\tilde{u}(x, 0, 0) = 0.$$

Hence, $\tilde{u}(x, 0, 0) \equiv 0$ in a sufficiently small neighbourhood of the origin ($x = y = z = 0$). It follows from (2.2) that

$$\tilde{v}(x, 0, 0) = \tilde{w}(x, 0, 0) \equiv 0$$

in the same neighbourhood. System (2.2) has the solution $x = \text{const}$, $\tilde{y} = \tilde{z} = 0$; substituting this into (2.3), we obtain

$$f(x, 0, 0) \equiv 0,$$

thereby proving the lemma.

We assume that system (1.1)-(1.3) is transformed in accordance with Lemma I and retain our former notation. We resolve the vector integral (1.3) relative to x

$$x = c + f^*(c, y, z), \tag{2.4}$$

where f^* is a vector-function analytic in c, y, and z ($f^*(c, 0, 0) = 0$) containing no linear terms for $c = 0$.

Substitution of (2.4) into (1.1) and (1.2) yields a vector system that is a Lyapunov system

$$\frac{dy}{dt} = \lambda J_2 y + v^*(c, y, z),$$
$$\frac{dz}{dt} = Bz + w^*(c, y, z), \tag{2.5}$$

and a scalar integral

$$(y, y) + \psi^*(c, y, z) = \gamma^*, \tag{2.6}$$

where the functions v^*, w^*, and ψ^* analytically depend on c, y, and z and vanish for $y = z = 0$ (by Lemma I), and the constant $\gamma^* = \gamma_0 - \psi(c, 0, 0)$. The vector-functions v^* and w^* contain no linear terms for $c = 0$.

The following lemma holds for the function ψ^*:

Lemma II. *Given a vector* $\mathbf{c}$ *in integral* (1.3) *of sufficiently small norm, the function* ψ^* *in* (2.6) *contains no terms of power one in* $\mathbf{y}$ *and* $\mathbf{z}$.

Proof. Let

$$\psi^* = (\mathbf{a},\ \mathbf{y}) + (\mathbf{b},\ \mathbf{z}) + (2),$$

where the terms of power higher than one in $\mathbf{y}$ and $\mathbf{z}$ are omitted, and the vectors $\mathbf{a}$ and $\mathbf{b}$ are analytic functions of $\mathbf{c}$ and vanish for $\mathbf{c} = \mathbf{0}$. Let us differentiate integral (2.6); by virtue of system (2.5),

$$(\mathbf{a},\ \lambda \mathbf{J}_2 \mathbf{y}) + (\mathbf{b},\ \mathbf{B}\mathbf{z}) + (2) = 0. \tag{2.7}$$

The determinant of the homogeneous system (2.7) with respect to the $2 + m$ components of the vectors $\mathbf{a}$ and $\mathbf{b}$ is a continuous function of $\mathbf{c}$ and becomes $\lambda^2 \det \mathbf{B} \neq \mathbf{0}$ for $\mathbf{c} = \mathbf{0}$. Hence, $\mathbf{c} = \mathbf{0}$ entails $\mathbf{a} = \mathbf{b} = \mathbf{0}$. Owing to continuity, these equalities must also hold for the vector $\mathbf{c}$ sufficiently small in norm.

The lemma is therefore proved.

The periodic solutions of a Lyapunov system (2.5) can be determined by the method suggested by Lyapunov [108a] and elaborated by Malkin (see [111a, b]). Other aspects of system (2.5) are discussed in Chapter III.

APPLICATION OF THE THEORY OF NORMAL FORMS TO OSCILLATION PROBLEMS

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CHAPTER V

ELEMENTS OF THE THEORY OF NORMAL FORMS OF REAL AUTONOMOUS SYSTEMS OF ORDINARY DIFFERENTIAL EQUATIONS

The problem of reducing an autonomous system of ordinary differential equations to its simplest, normal form by means of a change of variables was formulated by Poincaré [149a] and later developed by Lyapunov [108a] and others. The results of the most general character were obtained by Brjuno [238j]; the relevant references can be found in his publication. Sections 1 and 2 of this chapter describe the problem on the basis of Brjuno's results, but adjust the presentation to the problems of nonlinear oscillations governed by the systems of equations in question, thus avoiding the most general approach and the most general situations. Some of the proofs in Section 2 are omitted.

§ 1. Introductory Information

1.1. Statement of the problem. We consider systems of the type

$$\frac{dx_\nu}{dt} = \lambda_\nu x_\nu + \Phi_\nu(\mathrm{x}) \quad (\nu = 1, \ldots, n), \tag{1.1}$$

where x is a vector with the components $x_1, \ldots, x_n$; $\Phi_\nu(\mathrm{x})$ are analytic, in some neighbourhood of zero, functions whose expansions begin with terms of power not lower than two; λ_ν are either real or complex conjugate quantities; in the latter case $\lambda_{\nu'} = \overline{\lambda}_{\nu''}$, $x_{\nu'} = \overline{x}_{\nu''}$, $\Phi_{\nu'} = \overline{\Phi}_{\nu''}$. We thus assume that the initial real system is transformed in such a manner that its linear part is reduced to the Jordan form; it is assumed furthermore that this form is *diagonal*. By virtue of the Weierstrass theorem (see, for example [80], Sec. I, 1.14) this last condition holds for oscillations in conservative and quasiconservative systems.

The problem consists in reducing system (1.1) by means of a reversible (though not necessarily single-valued) transformation

$$x_\nu = y_\nu + \tilde{x}_\nu (y_1, \ldots, y_n) \quad (\nu = 1, \ldots, n, \tilde{x}_\nu (0, \ldots, 0) = 0) \quad (1.2)$$

to the simplest normal form (expansions $\tilde{x}_\nu$ begin with terms of power not lower than two). In particular, Brjuno [238a] introduced the following notation for the transformed system

$$\frac{dy_\nu}{dt} = \lambda_\nu y_\nu + y_\nu g_\nu (\mathbf{y}) \equiv \lambda_\nu y_\nu + y_\nu \sum_{\mathbf{Q} \in \mathfrak{M}_\nu} g_{\nu\mathbf{Q}} \mathbf{y}^{\mathbf{Q}} \quad (1.3)$$

$$(\mathbf{y} = (y_1, \ldots, y_n)^\tau, \quad \nu = 1, \ldots, n),$$

where $\mathbf{Q} = (q_1, \ldots, q_n)^\tau$ is a vector with integer components, $\mathbf{y}^{\mathbf{Q}} = y_1^{q_1} \ldots y_n^{q_n}$; and $g_{\nu\mathbf{Q}}$ are the coefficients to be found. The set $\mathfrak{M}_\nu$ of $\mathbf{Q}$ for the νth equation is

$$q_1, \ldots, q_{\nu-1}, q_{\nu+1}, \ldots, q_n \geqslant 0, \quad q_\nu \geqslant -1, \quad \sum_1^n q_j \geqslant 1 \quad (1.4)$$

(the reader should pay special attention to the multiplier y_ν in the νth equation). We introduce the symbol $\mathfrak{M} = \mathfrak{M}_1 \cup \mathfrak{M}_2 \cup \ldots \cup \mathfrak{M}_n$.

Normal forms are defined in the following theorem, which we formulate in terms of this subsection.

1.2. The Fundamental Brjuno theorem ([238j], Ch. I, § 1, I). *There exists a reversible transformation* (1.2) *of system* (1.1) *into system* (1.3) *such that, given this representation of* (1.3), *the* $g_{\nu\mathbf{Q}}$*'s can be nonvanishing only for those* $\mathbf{Q}$ *that satisfy the resonant equation*

$$(\mathbf{\Lambda}, \mathbf{Q}) \equiv \lambda_1 q_1 + \ldots + \lambda_n q_n = 0. \quad (2.1)$$

Here $\mathbf{\Lambda} = (\lambda_1, \ldots, \lambda_n)^\tau$ is a vector formed from the diagonal elements of the linear part of system (1.3) (and (1.1)). System (1.3) with this property is called a *normal form.*

Proof. Following Brjuno, let us prove that there exists a formal transformation (*normalizing transformation*)

$$x_\nu = y_\nu [1 + h_\nu (\mathbf{y})], \quad h_\nu (\mathbf{y}) = \sum_{\mathbf{Q} \in \mathfrak{M}_\nu} h_{\nu\mathbf{Q}} \mathbf{y}^{\mathbf{Q}} \quad (\nu = 1, \ldots, n) \quad (2.2)$$

of system

$$\frac{dx_\nu}{dt} = \lambda_\nu x_\nu + x_\nu f_\nu (\mathbf{x}), \quad f_\nu (\mathbf{x}) = \sum_{\mathbf{Q} \in \mathfrak{M}_\nu} f_{\nu\mathbf{Q}} \mathbf{x}^{\mathbf{Q}} \quad (\nu = 1, \ldots, n) \quad (2.3)$$

into system

$$\frac{dy_\nu}{dt} = \lambda_\nu y_\nu + y_\nu g_\nu (\mathbf{y}), \quad g_\nu (\mathbf{y}) = \sum_{\substack{\mathbf{Q} \in \mathfrak{M}_\nu \\ (\mathbf{\Lambda}, \mathbf{Q}) = 0}} g_{\nu\mathbf{Q}} \mathbf{y}^{\mathbf{Q}} \quad (\nu = 1, \ldots, n) \quad (2.4)$$

such that $g_{\nu Q} = 0$ if $(\Lambda, Q) \neq 0$, and $h_{\nu Q}$ for $(\Lambda, Q) = 0$ can be arbitrary (this is the reason for possible nonuniqueness); the remaining $h_{\nu Q}$ and $g_{\nu Q}$ are then single-valued. Here $y_\nu h_\nu(\mathbf{y})$, $x_\nu f_\nu(\mathbf{x})$, and $y_\nu g_\nu(\mathbf{y})$ are power series containing no zero- and first-power terms. Transformation (2.2) converts (2.3) into (2.4) if the series are formally equal in $y_1, \ldots, y_n$

$$\sum_{l=1}^{n} \frac{\partial [y_\nu (1+h_\nu)]}{\partial y_l} (\lambda_l y_l + y_l g_l) = \lambda_\nu y_\nu (1+h_\nu)$$

$$+ y_\nu (1+h_\nu) f_\nu (y_1(1+h_1), \ldots, y_n(1+h_n)) \quad (\nu = 1, \ldots, n).$$

After obvious transformations we obtain

$$y_\nu g_\nu + y_\nu \sum_{l=1}^{n} \frac{\partial h_\nu}{\partial y_l} \lambda_l y_l = -y_\nu g_\nu h_\nu - y_\nu \sum_{l=1}^{n} \frac{\partial h_\nu}{\partial y_l} y_l g_l$$

$$+ y_\nu (1+h_\nu) f_\nu (y_1(1+h_1), \ldots, y_n(1+h_n)) \quad (\nu = 1, \ldots, n). \qquad (2.5)$$

The coefficients of $y_\nu \mathbf{y}^Q$ in the νth equality of (2.5) are

$$g_{\nu Q} + (\Lambda, Q) h_{\nu Q}$$

$$= -\sum_{P+R=Q} h_{\nu P} g_{\nu R} - \sum_{l=1}^{n} \sum_{P+R=Q} h_{\nu P} p_l g_{lR} + \{(1+h_\nu) f_\nu\}_Q \qquad (2.6)$$

$$(Q \in \mathfrak{M}_\nu;\ \nu = 1, \ldots, n),$$

where the last term denotes the coefficient of $\mathbf{y}^Q$ in the series $(1 + h_\nu) f_\nu (y_1(1 + h_1), \ldots, y_n(1 + h_n))$.

The set of n-dimensional real vectors becomes completely ordered if the following procedure is employed: the vector $\mathbf{P}$ *precedes* the vector $\mathbf{Q}$ ($\mathbf{P} < \mathbf{Q}$) if the first nonvanishing difference in the sequence

$$\sum_1^n p_j - \sum_1^n q_j, \quad p_1 - q_1, \ldots, p_{n-1} - q_{n-1}$$

is negative. Obviously, only a finite number of vectors from $\mathfrak{M}$ precede $\mathbf{Q} \in \mathfrak{M}$. Note that only such h_{jP} and g_{jR} ($j = 1, \ldots, n$) enter the right-hand side of (2.6) for which $\mathbf{P}$ and $\mathbf{R}$ precede $\mathbf{Q}$. This is true for the first and second terms in the right-hand side of (2.6) because the subscripts there include only those $\mathbf{P}$ and $\mathbf{R}$ for which $\sum p_j + \sum r_j = \sum q_j$ and $\sum p_j > 0$, $\sum q_j > 0$, and therefore $\sum p_j < \sum q_j$ and $\sum r_j < \sum q_j$. Finally, $\{(1+h_\nu) f_\nu\}_Q$ contain only those h_{jP} for which $\sum p_j < \sum q_j$ because the series $x_\nu f_\nu(\mathbf{x})$ has no linear terms. Equalities (2.5) are satisfied if

(a) $g_{\nu Q} = 0$; $h_{\nu Q} = (\Lambda, Q)^{-1} c_{\nu Q}$ for $(\Lambda, Q) \neq 0$,

(b) $g_{\nu Q} = c_{\nu Q}$; $h_{\nu Q}$ is arbitrary for $(\Lambda, Q) = 0$, $Q \in \mathfrak{M}_\nu$ ($\nu = 1, \ldots, n$). Here $c_{\nu Q}$ stands for the right-hand side of (2.6). As a result, $g_{\nu Q}$ and $h_{\nu Q}$ are determined from Q in the manner described, thus proving the theorem.

Remark. As stipulated by (b), only (2.1) results in nonuniqueness of transformation (2.2). If for $q_1 + \ldots + q_n = m$ (2.1) holds only for a finite number of values of the initial parameters of the system, then it is logical that $h_{\nu Q}$ be based, if possible, on continuity.

1.3. The Poincaré theorem [149a]. *If* $\lambda_1, \ldots, \lambda_n$ *in system* (1.1) *satisfy the conditions*

(1)
$$\lambda_\nu \neq \sum_{j=1}^{n} p_j \lambda_j \quad (\nu = 1, \ldots, n), \tag{3.1}$$

for any nonnegative integer p_j *for which* $p_1 + \ldots + p_n \geqslant 2$, *and*

(2) *on the complex plane* λ *there exists a straight line* H *passing through zero such that all the points* $\lambda_1, \ldots, \lambda_n$ *lie on one side of* H,

then there exists a unique reversible and analytic in some neighbourhood of zero transformation (1.2), *converting system* (1.1) *into*

$$\frac{dy_\nu}{dt} = \lambda_\nu y_\nu \quad (\nu = 1, \ldots, n). \tag{3.2}$$

Proof. The existence of a unique formal transformation of (1.2) (which can also be written in the form (2.2)) to the normal form (3.2) follows from the theorem of Subsection 1.2. We write (Λ, Q), using (1.4), as

$$(\Lambda, Q) \equiv \sum_{1}^{n} q_j \lambda_j = -\lambda_\nu + \sum_{1}^{n} p_j \lambda_j \quad (\nu = 1, \ldots, n),$$

where $p_\nu = q_\nu + 1$, and the remaining $p_j = q_j$. The numbers $p_1, \ldots, p_n$ are nonnegative and $p_1 + \ldots + p_n \geqslant 2$. By condition (3.1), $(\Lambda, Q) \neq 0$, and accordingly the fundamental Brjuno theorem states that all $g_{\nu Q}$ vanish and all $h_{\nu Q}$ are single-valued (see condition (a) at the end of Subsection 1.2).

This completes the proof; the reader will find an analysis of the convergence of transformation (1.2) in Subsection 2.4 (see also [149a]; [139], 1st edition).

Remark. In the real-variable case discussed above, the spectrum $\lambda_1, \ldots, \lambda_n$ is symmetric with respect to the real axis. Therefore, condition (2) is satisfied not only by H but by a straight line $\overline{H}$ symmetric with respect to the real axis, and thus by the imaginary axis of the complex plane λ as well. Consequently, it is sufficient to check condition (2) for the imaginary axis.

§ 2. Additional Information

2.1. Some properties of normalizing transformations. At the end of the proof of the fundamental Brjuno theorem (Subsection 1.2) we mentioned that the coefficients $h_{\nu\mathbf{Q}}$ of the normalizing transformation (1, 2.2) are single-valued for *nonresonant terms*, that is, for $(\Lambda, \mathbf{Q}) \neq 0$, and can be chosen arbitrarily for *resonant terms*, that is, for $(\Lambda, \mathbf{Q}) = 0$. At the same time, although the structure of the normal form is fixed by the established numbering of the variables, its coefficients $g_{\nu\mathbf{Q}}$ depend on the choice of coefficients for the normalizing transformation. It is logical to assume that subsequent transformations of the variables, if carried out for resonant terms only, will transform one normal form into another. This clarifies the meaning of the Brjuno theorem ([238j], Ch. I, § 1, II):

If a transformation

$$y_\nu = z_\nu\,[1 + d_\nu(\mathbf{z})] \quad (\nu = 1, \ldots, n), \tag{1.1}$$

$$\mathbf{z} = \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix}, \quad d_\nu(\mathbf{z}) = \sum_{\mathbf{Q} \in \mathfrak{M}_\nu} d_{\nu\mathbf{Q}} \mathbf{z}^{\mathbf{Q}}$$

converts a normal form (1, 2.4) *into a normal form of the same structure, then* $d_{\nu\mathbf{Q}} = 0$ for $(\Lambda, \mathbf{Q}) \neq 0$, *that is, transformation* (1.1) *operates only on resonant terms* (*i.e. those for which* $(\Lambda, \mathbf{Q}) = 0$).

We define a transformation $\xi_\nu^0(\mathbf{y})$ as

$$\xi_\nu^0(\mathbf{y}) = y_\nu \Bigg[1 + \sum_{\substack{\mathbf{Q} \in \mathfrak{M}_\nu \\ (\Lambda, \mathbf{Q}) = 0}} h_{\nu\mathbf{Q}} \mathbf{y}^{\mathbf{Q}}\Bigg] \quad (\nu = 1, \ldots, n), \tag{1.2}$$

and consider it as the "arbitrary" part of the normalizing transformation (1, 2.2). We also assume that series (1.2) are convergent. This can be achieved, for instance, by setting all $h_{\nu\mathbf{Q}} = 0$ in (1.2), that is, by choosing

$$\xi_\nu^0(\mathbf{y}) \equiv y_\nu \quad (\nu = 1, \ldots, n). \tag{1.3}$$

As follows from the third Brjuno theorem ([238j], Ch. I, § 1, III), the convergence (or divergence) of any normalizing transformation (1, 2.2) is a corollary of the convergence (or divergence) of one such normalizing transformation (1, 2.2), provided series (1.2) are convergent.

2.2. Classification of normal forms; integrable normal forms. We consider the case when in (1, 1.1)

$$\operatorname{Re} \lambda_j \leqslant 0 \quad (j = 1, \ldots, n); \tag{2.1}$$

in other words, the complex plane λ contains no points $\lambda_1, \ldots, \lambda_n$ to the right of the imaginary axis. We denote

$$\lambda_j = -\mu_j + i\nu_j \quad (i = \sqrt{-1};\ j = 1, \ldots, n)$$

and assume that there are l $(0 \leqslant l \leqslant n)$ pure imaginary adjoint or zero eigenvalues of the matrix of the linear part of the initial system. We now number the variables so that

$$0 = \mu_1 = \ldots = \mu_l < \mu_{l+1} \leqslant \ldots \leqslant \mu_n. \tag{2.2}$$

We introduce the vectors

$$\mathbf{M} = \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_n \end{pmatrix}, \quad \mathbf{N} = \begin{pmatrix} \nu_1 \\ \vdots \\ \nu_n \end{pmatrix},$$

then

$$\mathbf{\Lambda} = -\mathbf{M} + i\mathbf{N}. \tag{2.3}$$

Let $\mathbf{V} = (v_1, \ldots, v_n)^\tau$ be an n-dimensional vector; we denote an l-dimensional vector by $\mathbf{V}' = (v_1, \ldots, v_l)^\tau$ and an $n-l$-dimensional vector by $\mathbf{V}'' = (v_{l+1}, \ldots, v_n)^\tau$. The inequality $\mathbf{V} \geqslant \mathbf{0}$ signifies that $v_1 \geqslant 0, \ldots, v_n \geqslant 0$. For instance, from (2.2) we obtain $\mathbf{M}' = \mathbf{0}$ and $\mathbf{M}'' > \mathbf{0}$.

The Brjuno theorem ([238j], Ch. I, § 2, II). *Under the previous assumptions, a normal form can be written as*

$$\dot{y}_j = \lambda_j y_j + \psi_j \quad (j = 1, \ldots, l), \tag{2.4}$$

$$\dot{y}_h = \lambda_h y_h + \sum_{j=l+1}^{n} b_{hj} y_j + \sum b_{hq_{l+1}\ldots q_{h-1}} y_{l+1}^{q_{l+1}} \ldots y_{h-1}^{q_{h-1}} \tag{2.5}$$

$$(h = l+1, \ldots, n).$$

Here ψ_j, b_{hj}, *and* $b_{hq_{l+1}\ldots q_{h-1}}$ *are power series in* $y_1, \ldots, y_l$; $b_{hh} \equiv 0$. The first sum in (2.5) is taken over those $j > l$ for which (2.7) is satisfied, and the second sum over all the integers $q_{l+1}, \ldots, q_{h-1}$ for which (2.8) is satisfied.

Proof. In the normal form (1, 2.4), $g_{\nu\mathbf{Q}} \neq 0$ only if $\mathbf{Q} \in \mathfrak{M}_\nu$, $(\mathbf{\Lambda}, \mathbf{Q}) = 0$. This last equation (resonant equation) is equivalent, owing to (2.2) and (2.3), to a system of two equations

$$(\mathbf{N}, \mathbf{Q}) = 0, \quad (\mathbf{M}, \mathbf{Q}) \equiv q_{l+1}\mu_{l+1} + \ldots + q_n\mu_n = 0. \tag{2.6}$$

In order to emphasize the number of the equation of the normal form to which $\mathbf{Q}$ corresponds, we give $\mathbf{Q}$ an appropriate subscript. The second equation of (2.6) has only such solutions $\mathbf{Q} \in \mathfrak{M}_\nu$ that

$$\mathbf{Q}''_\nu = 0 \quad \text{if} \quad \nu \leqslant l.$$

Solutions of the second equation of (2.6) for $\nu > l$ (if they exist) are such that

$$\mathbf{Q}''_\nu = \mathbf{e}_j - \mathbf{e}_\nu \quad \text{if} \quad \mu_j = \mu_\nu, \tag{2.7}$$

or

$$\mathbf{Q}''_\nu = \sum_{j=l+1}^{m} q_j \mathbf{e}_j - \mathbf{e}_\nu \tag{2.8}$$

$$\left(q_j \geqslant 0,\ l+1 < m < \nu,\ \mu_m < \mu_\nu;\ \sum q_j \mu_j = \mu_\nu\right).$$

Finally, if $\nu \leqslant l$, then $q_{l+1} = \ldots = q_n = 0$, that is, $y_{l+1}, \ldots, y_n$ in ψ_j (see (2.4)) are to the zero power. This means that ψ_j are independent of $\mathbf{y}''$. If $\nu > l$, then $\mathbf{Q}''_\nu$ are of the form (2.7) or (2.8). The terms given in (2.5) correspond to these $\mathbf{Q}$, which completes the proof of the theorem.

Let us analyze several special cases of the above theorem.

(a) Let $l = 0$ and $0 < \mu_1 < \ldots < \mu_n$. Then subsystem (2.4) cancels out, the coefficients of the series in (2.5) are constants, and the normal form becomes triangular

$$\dot{y}_\nu = \lambda_\nu y_\nu + \sum b_{\nu q_1 \ldots q_{\nu-1}} y_1^{q_1} \ldots y_{\nu-1}^{q_{\nu-1}} \tag{2.9}$$
$$(\nu = 1, \ldots, n),$$

where the sum is taken over all the nonnegative integers $q_1, \ldots, q_{\nu-1}$ such that

$$\lambda_\nu = q_1\lambda_1 + \ldots + q_{\nu-1}\lambda_{\nu-1} \tag{2.10}$$
$$(q_1 + \ldots + q_{\nu-1} \geqslant 2;\quad \nu = 1, \ldots, n).$$

This normal form was derived by Dulac [253]. Equations (2.10) have a finite number of solutions (and no solutions in the hypothesis of the Poincaré theorem of Subsection 1.3). Therefore the right-hand sides of the normal form (2.9) are polynomials. Equations (2.9) are successively solved in quadratures.

(b) Let $\lambda_1 = \ldots = \lambda_l = 0$. The integration of the normal form (2.4) or (2.5) is carried out in two steps: solution of the lth-order system (2.4), and successive quadratures, as in case (a).

(c) The case of m pairs of pure imaginary $\left(0 < m \leqslant \frac{1}{2} l\right)$ and $l - 2m$ vanishing eigenvalues of λ_j (see [238j], Ch. I, § 2, II, p. 151).

(d) If $l = n$, the theorem yields no simplification.

2.3. Concept of power transformations. We wish to discuss the possibility of lowering the order of the normal form (1, 2.4) by means of the birational transformations

$$z_\nu = y_1^{\alpha_{\nu 1}} \ldots y_n^{\alpha_{\nu n}} \quad (\nu = 1, \ldots, n) \tag{3.1}$$
$$(\alpha_{\nu j} \text{ are real, } \mathbf{A} = \|\alpha_{\nu j}\|_1^n,\ \det \mathbf{A} \neq 0).$$

Each coefficient $g_{\nu Q} \neq 0$ in (1, 2.4) has corresponding to it a point $\mathbf{Q}$ of the n-dimensional integer lattice $\mathfrak{M} = \mathfrak{M}_1 \cup \mathfrak{M}_2 \cup \ldots \cup \mathfrak{M}_n$. The set of these points is denoted by $\mathfrak{D}(\mathbf{G_Q})$, that is, the set $\mathfrak{D}(\mathbf{G_Q})$ in $\mathfrak{M}$ is defined by

$$(\Lambda, \mathbf{Q}) = 0, \quad \mathbf{Q} \in \mathfrak{M}_\nu, \quad \mathbf{G_Q} \equiv (g_{1Q}, \ldots, g_{nQ})^\tau \neq \mathbf{0}.$$

We denote

$$\ln \mathbf{y} = \begin{pmatrix} \ln y_1 \\ \vdots \\ \ln y_n \end{pmatrix}, \quad \ln \mathbf{z} = \begin{pmatrix} \ln z_1 \\ \vdots \\ \ln z_n \end{pmatrix}$$

and write system (1, 2.4) and transformation (3.1) in the form

$$\frac{d \ln \mathbf{y}}{dt} = \Lambda + \sum_{\mathbf{Q} \in \mathfrak{D}(\mathbf{G_Q})} \mathbf{G_Q} \exp(\ln \mathbf{y}, \mathbf{Q}),$$

$$\ln \mathbf{z} = \mathbf{A} \ln \mathbf{y}.$$

The term $\mathbf{y^Q}$ is transformed by (3.1) into

$$\mathbf{y^Q} = \exp(\ln \mathbf{y}, \mathbf{Q}) = \exp(\mathbf{A}^{-1} \ln \mathbf{z}, \mathbf{Q}) = \exp(\ln \mathbf{z}.\ \mathbf{A}^{-1*}\mathbf{Q}) = \mathbf{z}^{\mathbf{A}^{-1*}\mathbf{Q}}.$$

Finally we obtain

$$\frac{d \ln \mathbf{z}}{dt} = \mathbf{A} \frac{d \ln \mathbf{y}}{dt} = \mathbf{A}\Lambda + \sum_{\mathbf{Q} \in \mathfrak{D}(\mathbf{G_Q})} \mathbf{A}\mathbf{G_Q}\mathbf{z}^{\mathbf{A}^{-1*}\mathbf{Q}}.$$

We denote

$$\mathbf{P} = \mathbf{A}^{-1*}\mathbf{Q}, \quad \widetilde{\mathbf{G}}_\mathbf{P} = \mathbf{A}\mathbf{G_Q} \tag{3.2}$$

and represent this system in the form

$$\frac{d \ln \mathbf{z}}{dt} = \mathbf{A}\Lambda + \sum_{\mathbf{P} \in \mathfrak{D}(\widetilde{\mathbf{G}}_\mathbf{P})} \widetilde{\mathbf{G}}_\mathbf{P} \mathbf{z^P}. \tag{3.3}$$

The set $\mathfrak{D}(\widetilde{\mathbf{G}}_\mathbf{P})$ of points $\mathbf{P}$ for which $\widetilde{\mathbf{G}}_\mathbf{P} \neq \mathbf{0}$ is obtained from $\mathfrak{D}(\mathbf{G_Q})$ by the linear transformation (3.2)

$$\mathfrak{D}(\widetilde{\mathbf{G}}_\mathbf{P}) = \mathbf{A}^{-1} \mathfrak{D}(\mathbf{G_Q}).$$

System (3.3) is not necessarily of the same type as the initial one (we mean analyticity in some neighbourhood of zero). But if it is analytic in some neighbourhood of zero, and the initial system is a normal form, then (3.3) is also a normal form (for the integer matrix $\mathbf{A}$).

The possibility of lowering the order of the form is established by the Brjuno theorem ([238j], Ch. I, § 2, I):

Let d be the number of linearly independent $\mathbf{Q} \in \mathfrak{M}$ *satisfying equation* $(\Lambda, \mathbf{Q}) = 0$. *There exists a birational transformation* (3.1),

with an integer unimodular matrix $\mathbf{A}$ ($\alpha_{\nu j}$ *are integers*, $\det \mathbf{A} = \pm 1$) *that transforms the normal form* (1, 2.4) *to* (3.3). *The first d equations of this system form a system of order d, the remaining equations being reducible to quadratures.*

Methods for efficiently constructing $\mathbf{A}$ can be found in [238c].

2.4. The Brjuno theorem on convergence and divergence of normalizing transformations. In the normalizing transformation (1, 2, 2) we single out series (1.2) in resonant terms. Following ([238j], § 0, II), we formulate conditions ω, $\overline{\omega}$, and $\mathbf{A}$. We assume

$$\omega_k = \min |(\Lambda, \mathbf{Q})| \text{ in } (\Lambda, \mathbf{Q}) \neq 0,\ q_1 + \ldots + q_n < 2^k. \quad (4.1)$$

Condition ω:

$$-\sum_{k=1}^{\infty} \frac{\ln \omega_k}{2^k} < \infty. \quad (4.2)$$

Condition $\overline{\omega}$:

$$\overline{\lim_{k\to\infty}} - \frac{\ln \omega_k}{2^k} < \infty.$$

For condition A, we recall that inequalities (2.1) were assumed satisfied in each case. As in Subsection 2.1, we denote those elements of λ_ν that lie on the imaginary axis by $\lambda_1, \ldots, \lambda_l$ ($0 \leqslant l \leqslant n$).

Condition A′: *There exists a power series a* $(y_1, \ldots, y_l)$ *such that in* (2.4)

$$\psi_j \equiv \lambda_j y_j a \quad (j = 1, \ldots, l).$$

In the case of condition A″, we distinguish two situations:

(1*) the numbers $\lambda_1, \ldots, \lambda_l$ are pairwise commensurable.

(1**) $\lambda_1, \ldots, \lambda_l$ include at least one pair of incommensurable elements.

Condition A″: *If* Λ *is covered by case* (1*), *then series* b_{hj} ($h, j = l+1, \ldots, n$) *in* (2.5) *are arbitrary, and if* Λ *is covered by case* (1**), *then there exist power series* $a_{l+1}, \ldots, a_n$ *in* $y_1, \ldots, y_l$ *such that*

(a) If $\mathbf{Q} \in \mathfrak{M}$, $(\mathbf{M}, \mathbf{Q}) = 0$, then $q_{l+1}a_{l+1} + \ldots + q_n a_n \equiv 0$.

(b) *The* $(n-l) \times (n-l)$ *matrix*

$$\mathbf{B} = \| b_{hj} - \delta_{hj}(\lambda_h a + a_h) \|_{l+1}^{n}$$

(δ_{hj} is the Kronecker delta) *is nilpotent, that is,* $\mathbf{B}^{n-l} \equiv \mathbf{0}$.

Condition A (in the case considered above when inequalities (2.1) are satisfied) *is the simultaneous fulfillment of conditions* A′ *and* A″.

The Brjuno theorem on convergence and divergence of normalizing transformations ([238j], Ch. II, III).

(1) *If for a convergent system* (1, 1.1) Λ *satisfies condition* ω, *and a normal form* (1, 2.4) *satisfies condition* A, *then the transformation* (1, 2.2) *converting* (1, 1.1) *to* (1, 2.4) *is convergent if and only if all series* (1.2) *are convergent in some neighbourhood of zero.*

(2) *If one or both conditions* $\overline{\omega}$ *and* A *are violated for the normal form* (1, 2.4), *then there exists a convergent system* (1, 1.1) *for which system* (1, 2.4) *is a normal form and for which each transformation to normal form is divergent.*

To illustrate this, we return to the Poincaré theorem (Subsection 1.3) for real systems, that is, when H is the imaginary axis. Obviously, in this case condition A is trivial (automatically satisfied) and

$$\omega_k = \min \mid q_1\lambda_1 + \ldots + q_n\lambda_n \mid > \min \mid \operatorname{Re} \lambda_\nu \mid = \text{const} > 0$$

(see (4.1)); as a result, the left-hand side of inequality (4.2) is also a constant, and condition ω is thus satisfied. Moreover, in the hypothesis of the Poincaré theorem, a normalizing transformation is also single-valued, so that according to statement (1) of the above theorem this transformation is convergent in some neighbourhood of its zero values.

§ 3. Practical Calculation of Coefficients of Normalizing Transformation and Normal Form

3.1. Fundamental identities. We assume that an oscillatory system is described by a real autonomous system of nth-order differential equations. We also assume that this system is reduced to diagonal form and that its right-hand side has complex coefficients and is analytic in some neighbourhood of its zero values

$$\frac{dx_\nu}{dt} = \lambda_\nu x_\nu + \sum a^\nu_{jh} x_j x_h + \sum b^\nu_{jhk} x_j x_h x_k + \ldots \quad (\nu = 1, \ldots, n). \quad (1.1)$$

Here and henceforth a repeated subscript indicates summation and takes on the values $1, \ldots, n$; the coefficients are symmetrized, that is,

$$a^\nu_{hj} = a^\nu_{jh}, \quad b^\nu_{\{jhk\}} = \text{id.} \quad (\nu, j, h, k = 1, \ldots, n)$$

and braces in $\{\alpha\beta \ldots \omega\}$ denote any permutation of the elements $\alpha, \beta, \ldots, \omega$.

According to the fundamental Brjuno theorem (see Subsection 1.2), there exists a reversible (though not necessarily single-valued, and in some cases divergent) normalizing transformation with, in general, complex coefficients

$$x_j = y_j + \sum \alpha^j_{lm} y_l y_m + \sum \beta^j_{lmp} y_l y_m y_p + \cdots \quad (1.2)$$

$$(\alpha^j_{ml} = \alpha^j_{lm}, \quad \beta^j_{\{lmp\}} = \text{id.}; \quad j, l, m, p = 1, \ldots, n)$$

that reduces system (1.1) to the normal form

$$\frac{dy_\nu}{dt}=\lambda_\nu y_\nu+y_\nu\sum_{(\Lambda,\ \mathbf{Q})=0} g_{\nu\mathbf{Q}}y_1^{q_1}\dots y_n^{q_n}\quad(\nu=1,\dots,n).\qquad(1.3)$$

Here Λ and $\mathbf{Q}$ are vectors with the components $\lambda_1,\dots,\lambda_n$ and $q_1,\dots,q_n$, respectively, the latter being integers

$$q_\nu\geqslant-1,\quad q_j\geqslant 0\quad(j\neq\nu),\quad q_1+\dots+q_n\geqslant 1,\qquad(1.4)$$

and $g_{\nu\mathbf{Q}}$ is a general notation for coefficients of a normal form. Summation in (1.3) involves only *resonant terms* satisfying the *resonant equation* (see (1, 2.1))

$$(\Lambda,\ \mathbf{Q})\equiv q_1\lambda_1+\dots+q_n\lambda_n=0.\qquad(1.5)$$

Let us symmetrize the coefficients of the normal form (1.3) and express them in the form

$$\frac{dy_\nu}{d\tau}=\lambda_\nu y_\nu+\sum\varphi_{lm}^\nu y_ly_m+\sum\chi_{lmp}^\nu y_ly_my_p+\dots\qquad(1.3\text{a})$$
$$(\varphi_{ml}^\nu=\varphi_{lm}^\nu,\quad\chi_{\{lmp\}}^\nu=\text{id.};\quad\nu,l,m,p=1,\dots,n).$$

The nonvanishing coefficients φ_{lm}^ν, χ_{lmp}^ν, ... in (1.3a) are of course defined by representation (1.3). Substitution of (1.2) transforms (1.1) to a normal form. By dropping the terms with powers above three, we obtain the formal identities (the derivatives with respect to τ are primed)

$$\begin{aligned}y_\nu'+\sum\alpha_{lm}^\nu(y_l'y_m+y_ly_m')+\sum\beta_{lmp}^\nu(y_l'y_my_p+y_ly_m'y_p+y_ly_my_p')+\dots\\=\lambda_\nu y_\nu+\lambda_\nu\sum\alpha_{lm}^\nu y_ly_m+\lambda_\nu\sum\beta_{lmp}^\nu y_ly_my_p\\+\sum a_{jh}^\nu\left(y_j+\sum\alpha_{lm}^jy_ly_m\right)\left(y_h+\sum\alpha_{lm}^hy_ly_m\right)\\+\sum b_{jhk}^\nu y_jy_hy_k+\dots\quad(\nu=1,\dots,n),\end{aligned}$$

where dots stand for terms of powers not lower than four. By virtue of (1.3a), we obtain

$$\begin{aligned}\sum\varphi_{lm}^\nu y_ly_m+\sum\chi_{lmp}^\nu y_ly_my_p+\sum\alpha_{lm}^\nu\left[(\lambda_l+\lambda_m)y_ly_m+y_m\sum\varphi_{jp}^ly_jy_p\right.\\\left.+y_l\sum\varphi_{jp}^my_jy_p\right]+\sum\beta_{lmp}^\nu(\lambda_l+\lambda_m+\lambda_p)y_ly_my_p+\dots\\=\lambda_\nu\sum\alpha_{lm}^\nu y_ly_m+\lambda_\nu\sum\beta_{lmp}^\nu y_ly_my_p+\sum a_{jh}^\nu y_jy_h+\sum a_{jh}^\nu\alpha_{lm}^hy_jy_ly_m\\+\sum a_{jh}^\nu\alpha_{lm}^jy_hy_ly_m+\sum b_{jhk}^\nu y_jy_hy_k+\dots\quad(\nu=1,\dots,n).\end{aligned}$$

By changing the summation indices and symmetrizing the coefficients in the sums, we derive the fundamental identities

$$\sum \varphi^{\nu}_{lm} y_l y_m + \sum \chi^{\nu}_{lmp} y_l y_m y_p$$

$$+\frac{2}{3}\sum \left(\alpha^{\nu}_{jl}\varphi^{j}_{mp} + \alpha^{\nu}_{jm}\varphi^{j}_{pl} + \alpha^{\nu}_{jp}\varphi^{j}_{lm}\right) y_l y_m y_p$$

$$+\sum (\lambda_l + \lambda_m - \lambda_\nu)\,\alpha^{\nu}_{lm} y_l y_m + \sum (\lambda_l + \lambda_m + \lambda_p - \lambda_\nu)\,\beta^{\nu}_{lmp} y_l y_m y_p + \ldots$$

$$= \sum a^{\nu}_{lm} y_l y_m + \sum b^{\nu}_{lmp} y_l y_m y_p$$

$$+\frac{2}{3}\sum \left(a^{\nu}_{jl}\alpha^{j}_{mp} + a^{\nu}_{jm}\alpha^{j}_{pl} + a^{\nu}_{jp}\alpha^{j}_{lm}\right) y_l y_m y_p + \ldots \qquad (1.6)$$

$$(\nu = 1, \ldots, n).$$

3.2. Computational alternative. We introduce the symbols

$$\Delta^{\nu}_{lm} = \begin{cases} 1 & \text{if} \quad \lambda_\nu = \lambda_l + \lambda_m, \\ 0 & \text{if} \quad \lambda_\nu \neq \lambda_l + \lambda_m; \end{cases}$$

$$\Delta^{\nu}_{lmp} = \begin{cases} 1 & \text{if} \quad \lambda_\nu = \lambda_l + \lambda_m + \lambda_p, \\ 0 & \text{if} \quad \lambda_\nu \neq \lambda_l + \lambda_m + \lambda_p \end{cases} \qquad (2.1)$$

$$(\nu,\ l,\ m,\ p = 1, \ldots, n).$$

The following *alternative* is true:

(1) Suppose that the values taken on by ν, l, m, and p (and by the real parameters of the initial oscillatory system on which λ_ν, λ_l, λ_m, and λ_p depend) are such that the expressions in parentheses in the fourth and fifth sums in the left-hand side of (1.6) are distinct from zero ($\Delta^{\nu}_{lm} = 0$ and $\Delta^{\nu}_{lmp} = 0$). Comparing the terms containing $y_l y_m$ in the left- and right-hand sides of the fundamental identities (1.6) and repeating this process for $y_l y_m y_p$, we notice that the corresponding term in the first and second sums on the left definitely vanish. Indeed, (1.3) yields the term containing $y_l y_m$ in the first sum

$$y_\nu \varphi^{\nu}_{lm} y_l y_m y_\nu^{-1}.$$

Under the assumption made at the beginning of (1), for this term we have $(\mathbf{\Lambda},\ \mathbf{Q}) = \lambda_l \cdot 1 + \lambda_m \cdot 1 + \lambda_\nu \cdot (-1) \neq 0$, and the first sum on the left-hand side of (1.6), according to (1.3), includes only the terms for which $(\mathbf{\Lambda},\ \mathbf{Q}) = 0$. The absence of the term containing $y_l y_m y_p$ in the second sum of (1.6) is established in a similar manner. Equating the coefficients of the quadratic terms in (1.6), we obtain

$$\alpha^{\nu}_{lm} = \frac{a^{\nu}_{lm}}{\lambda_l + \lambda_m - \lambda_\nu}, \qquad (2.2)$$

and repeating this operation for the cubic terms, we have

$$\beta^{\nu}_{lmp}=\frac{1}{\lambda_l+\lambda_m+\lambda_p-\lambda_\nu}\left\{b^{\nu}_{lmp}+\frac{2}{3}\sum_{j=1}^{n}[a^{\nu}_{jl}\alpha^{j}_{mp}+a^{\nu}_{jm}\alpha^{j}_{pl}+a^{\nu}_{jp}\alpha^{j}_{lm}\right.$$

$$\left.-(\alpha^{\nu}_{jl}\varphi^{j}_{mp}+\alpha^{\nu}_{jm}\varphi^{j}_{pl}+\alpha^{\nu}_{jp}\varphi^{j}_{lm})]\right\}. \qquad (2.3)$$

We wish to emphasize that expressions (2.2) and (2.3), as we have already stated, are valid for those values of ν, l, m, and p from $1, \ldots, n$ for which the denominators of the formulas do not vanish.

(2) Suppose that ν, l, m, and p (and the real parameters of the initial oscillatory system) are such that the expressions in parentheses in the fourth and fifth sums in (1.6) vanish ($\Delta^{\nu}_{lm}=1$ and $\Delta^{\nu}_{lmp}=1$). First, this means that the corresponding α^{ν}_{lm} and β^{ν}_{lmp} can be chosen arbitrarily (set to zero, for example) or determined by continuity from the values of the real parameters. This has been already mentioned (see condition (b)) at the end of the proof of the fundamental Brjuno theorem (Subsections 1.2 and 2.1). Second, the assumption made yields $(\mathbf{\Lambda}, \mathbf{Q})=0$. Comparison of the terms containing $y_l y_m$ and $y_l y_m y_p$ in the left- and right-hand sides of the fundamental identities (1.6) now yields the symmetrized coefficients of the normal form (1.3a), namely, φ^{ν}_{lm} and χ^{ν}_{lmp}. We obtain

$$\varphi^{\nu}_{lm}=\Delta^{\nu}_{lm}a^{\nu}_{lm} \quad (\nu, l, m=1, \ldots, n), \qquad (2.4)$$

$$\chi^{\nu}_{lmp}=\Delta^{\nu}_{lmp}\left\{b^{\nu}_{lmp}+\frac{2}{3}\sum_{j=1}^{n}[a^{\nu}_{jl}\alpha^{j}_{mp}+a^{\nu}_{jm}\alpha^{j}_{pl}+a^{\nu}_{jp}\alpha^{j}_{lm}\right.$$

$$\left.-(\alpha^{\nu}_{jl}\varphi^{j}_{mp}+\alpha^{\nu}_{jm}\varphi^{j}_{pl}+\alpha^{\nu}_{jp}\varphi^{j}_{lm})]\right\} \quad (\nu, l, m, p=1, \ldots, n). \qquad (2.5)$$

It should be emphasized that formulas (2.4) and (2.5), although derived for case (2), are valid for all values of the subscripts, namely, in case (1) by virtue of the notation of (2.1), these formulas yield zeros for the corresponding values of ν, l, m, and p.

Let us demonstrate now that if in case (2) $\alpha^{\nu}_{lm}=0$ $(\lambda_\nu=\lambda_l+\lambda_m)$*, then all the terms in parentheses in (2.5) vanish. We shall show, for instance, that $\alpha^{\nu}_{jl}\varphi^{j}_{mp}=0$ $(j=1, \ldots, n)$. We first assume that $\Delta^{j}_{mp}=0$; it then follows from (2.4) that $\varphi^{j}_{mp}=0$, so that our assertion is true. We now have to analyze only the case when $\Delta^{j}_{mp}=1$, that is,

$$\lambda_j=\lambda_m+\lambda_p. \qquad \text{(a)}$$

In case (2) $\Delta^{\nu}_{lmp}=1$, that is,

$$\lambda_\nu=\lambda_l+\lambda_m+\lambda_p. \qquad \text{(b)}$$

* Obviously, this choice results in $\Delta^{\nu}_{lm}\alpha^{\nu}_{lm}=0$ $(\nu, l, m=1, \ldots, n)$.

Subtracting (a) from (b), we obtain $\lambda_\nu = \lambda_j + \lambda_l$, that is, $\Delta^\nu_{jl} = 1$; our assumption stipulates that $\alpha^\nu_{jl} = 0$, so that again $\alpha^\nu_{jl}\varphi^j_{mp} = 0$ ($j = 1, \ldots, n$). The proofs for the remaining terms in parentheses are similar because these terms are obtained from the first terms by circular permutation of the subscripts.

Summarizing, *if all arbitrary quadratic coefficients of the normalizing transformation are chosen to be zero, that is,*

$$\alpha^\nu_{lm} = 0 \quad \textit{for} \quad \Delta^\nu_{lm} = 1,$$

or if the normal form contains no quadratic terms, formula (2.5) *is simplified to*

$$\chi^\nu_{lmp} = \Delta^\nu_{lmp}\left\{b^\nu_{lmp} + \frac{2}{3}\sum_{j=1}^{n}[a^\nu_{jl}\alpha^j_{mp} + a^\nu_{jm}\alpha^j_{pl} + a^\nu_{jp}\alpha^j_{lm}]\right\} \tag{2.6}$$

$$(\nu, l, m, p = 1, \ldots, n).$$

A general procedure for determining the coefficients of a normalizing transformation and a normal form is contained in the proof of the fundamental Brjuno theorem (Subsection 1.2). The method we have outlined above seems to us better suited for applications to oscillation problems; hence, the term "practical" in the section title.

3.3. Fundamental identities in general form and their transformation. Unification of notation in the initial diagonal system (1.1), the normalizing transformation (1.2), and the normal form (1.3a) yields

$$\frac{dx_\nu}{dt} = \lambda_\nu x_\nu + \sum a^\nu_{j_1 j_2} x_{j_1} x_{j_2} + \ldots + \sum a^\nu_{j_1 \ldots j_\varkappa} x_{j_1} \ldots x_{j_\varkappa} + \ldots \tag{3.1}$$

$$(\nu = 1, \ldots, n),$$

$$x_\nu = y_\nu + \sum \alpha^\nu_{j_1 j_2} y_{j_1} y_{j_2} + \ldots + \sum \alpha^\nu_{j_1 \ldots j_\varkappa} y_{j_1} \ldots y_{j_\varkappa} + \ldots \tag{3.2}$$

$$(\nu = 1, \ldots, n),$$

$$\frac{dy_\nu}{dt} = \lambda_\nu y_\nu + \sum \varphi^\nu_{j_1 j_2} y_{j_1} y_{j_2} + \ldots + \sum \varphi^\nu_{j_1 \ldots j_\varkappa} y_{j_1} \ldots y_{j_\varkappa} + \ldots \tag{3.3}$$

$$(\nu = 1, \ldots, n).$$

We recall that all the coefficients are symmetrized, that is, they remain unaltered by arbitrary permutations of subscripts

$$a^\nu_{\{j_1 \ldots j_\varkappa\}} = \text{id.}, \quad \alpha^\nu_{\{j_1 \ldots j_\varkappa\}} = \text{id.}, \quad \varphi^\nu_{\{j_1 \ldots j_\varkappa\}} = \text{id.} \tag{3.4}$$

$$(\nu = 1, \ldots, n, \quad \varkappa = 2, 3, \ldots).$$

The subscripts repeated twice denote summation from 1 to n, summation being independent of other subscripts.

We also use the concise notation

$$\frac{dx_\nu}{dt}=\sum_{\varkappa=1}^{\infty}\sum a^{\nu}_{j_1\ldots j_\varkappa}x_{j_1}\ldots x_{j_\varkappa}\quad(\nu=1,\ \ldots,\ n),\tag{3.1a}$$

$$x_\nu=\sum_{\varkappa=1}^{\infty}\sum \alpha^{\nu}_{j_1\ldots j_\varkappa}y_{j_1}\ldots y_{j_\varkappa}\quad(\nu=1,\ \ldots,\ n),\tag{3.2a}$$

$$\frac{dy_\nu}{dt}=\sum_{\varkappa=1}^{\infty}\sum \varphi^{\nu}_{j_1\ldots j_\varkappa}y_{j_1}\ldots y_{j_\varkappa}\quad(\nu=1,\ \ldots,\ n),\tag{3.3a}$$

where

$$a^{\nu}_{j}=\lambda_\nu\delta_{\nu j},\quad \alpha^{\nu}_{j}=\delta_{\nu j},\quad \varphi^{\nu}_{j}=\lambda_\nu\delta_{\nu j},\tag{3.5}$$

and $\delta_{\nu j}$ is the Kronecker delta; $\delta_{\nu\nu}=1$, $\delta_{\nu j}=0$ $(j\neq\nu)$. Substitution of (3.2) into (3.1), taking into account (3.3), yields the following formal identities (fundamental identities in general form)

$$\begin{aligned}
&\lambda_\nu y_\nu+\sum\varphi^{\nu}_{j_1j_2}y_{j_1}y_{j_2}+\ldots+\sum\varphi^{\nu}_{j_1\ldots j_\varkappa}y_{j_1}\ldots y_{j_\varkappa}+\\
&\ldots+\sum\alpha^{\nu}_{j_1j_2}(\dot{y}_{j_1}y_{j_2}+y_{j_1}\dot{y}_{j_2})+\ldots+\sum\alpha^{\nu}_{j_1\ldots j_\varkappa}(\dot{y}_{j_1}y_{j_2}\ldots y_{j_\varkappa}+\\
&\ldots+y_{j_1}\ldots y_{j_{\mu-1}}\dot{y}_{j_\mu}y_{j_{\mu+1}}\ldots y_{j_\varkappa}+\ldots+y_{j_1}\ldots y_{j_{\varkappa-1}}\dot{y}_{j_\varkappa})+\\
&\ldots=\lambda_\nu y_\nu+\lambda_\nu\sum\alpha^{\nu}_{j_1j_2}y_{j_1}y_{j_2}+\ldots+\lambda_\nu\sum\alpha^{\nu}_{j_1\ldots j_\varkappa}y_{j_1}\ldots y_{j_\varkappa}+\\
&\ldots+\sum a^{\nu}_{j_1j_2}\Big(\sum\alpha^{j_1}_{j^1_1}y_{j^1_1}+\ldots+\sum\alpha^{j_1}_{j^1_1\ldots j^1_\vartheta}y_{j^1_1}\ldots y_{j^1_\vartheta}+\\
&\ldots+\sum\alpha^{j_1}_{j^1_1\ldots j^1_{k-1}}y_{j^1_1}\ldots y_{j^1_{k-1}}+\ldots\Big)\Big(\sum\alpha^{j_2}_{j^2_1}y_{j^2_1}+\\
&\ldots+\sum\alpha^{j_2}_{j^2_1\ldots j^2_\vartheta}y_{j^2_1}\ldots y_{j^2_\vartheta}+\ldots+\sum\alpha^{j_2}_{j^2_1\ldots j^2_{k-1}}y_{j^2_1}\ldots y_{j^2_{k-1}}\Big)+\\
&\ldots+\sum a^{\nu}_{j_1\ldots j_\varkappa}\Big(\sum\alpha^{j_1}_{j^1_1}y_{j^1_1}+\ldots+\sum\alpha^{j_1}_{j^1_1\ldots j^1_\vartheta}y_{j^1_1}\ldots y_{j^1_\vartheta}+\\
&\ldots+\sum\alpha^{j_1}_{j^1_1\ldots j^1_{k-\varkappa+1}}y_{j^1_1}\ldots y_{j^1_{k-\varkappa+1}}+\ldots\Big)\times\\
&\ldots\times\Big(\sum\alpha^{j_\varkappa}_{j^\varkappa_1}y_{j^\varkappa_1}+\ldots+\sum\alpha^{j_\varkappa}_{j^\varkappa_1\ldots j^\varkappa_\vartheta}y_{j^\varkappa_1}\ldots y_{j^\varkappa_\vartheta}+\\
&\qquad\ldots+\sum\alpha^{j_\varkappa}_{j^\varkappa_1\ldots j^\varkappa_{k-\varkappa+1}}y_{j^\varkappa_1}\ldots y_{j^\varkappa_{k-\varkappa+1}}+\ldots\Big)+\ldots\\
&\qquad(\nu=1,\ \ldots,\ n).
\end{aligned}$$

The terms of these identities with variables to the kth power are, by (3.3),

$$\sum \varphi^{\nu}_{j_1\ldots j_k} y_{j_1}\ldots y_{j_k} + \sum_{\varkappa=2}^{k-1}\sum_{\mu=1}^{\varkappa}\sum_{j_1,\ldots,j_\varkappa}\alpha^{\nu}_{j_1\ldots j_\varkappa} y_{j_1}\ldots y_{j_{\mu-1}} y_{j_{\mu+1}}\ldots y_{j_\varkappa}$$
$$\times \sum_{j_1^\mu,\ldots,j^\mu_{k-\varkappa+1}} \varphi^{j_\mu}_{j_1^\mu\ldots j^\mu_{k-\varkappa+1}} y_{j_1^\mu}\ldots y_{j^\mu_{k-\varkappa+1}}$$
$$+\sum(\lambda_{j_1}+\ldots+\lambda_{j_k})\,\alpha^{\nu}_{j_1\ldots j_k} y_{j_1}\ldots y_{j_k} = \lambda_\nu \sum \alpha^{\nu}_{j_1\ldots j_k} y_{j_1}\ldots y_{j_k}$$
$$+\sum_{\varkappa=2}^{k-1}\sum_{i_1,\ldots,i_\varkappa} a^{\nu}_{i_1\ldots i_\varkappa}\sum_{\mu_1+\ldots+\mu_\varkappa=k}\sum_{j_1^1,\ldots,j^\varkappa_{\mu_\varkappa}} \alpha^{i_1}_{j_1^1\ldots j^1_{\mu_1}}\ldots \alpha^{i_\varkappa}_{j_1^\varkappa\ldots j^\varkappa_{\mu_\varkappa}}$$
$$\times y_{j_1^1}\ldots y_{j^1_{\mu_1}}\ldots y_{j_1^\varkappa}\ldots y_{j^\varkappa_{\mu_\varkappa}} + \sum a^{\nu}_{j_1\ldots j_k} y_{j_1}\ldots y_{j_k} \qquad (3.6)$$
$$(\nu = 1,\ \ldots,\ n),$$

where $\mu_1, \ldots, \mu_{k-1}$ are natural numbers.

We now compare the coefficients of $y_{j_1}\ldots y_{j_k}$, where $j_1, \ldots, j_k$ represent any sequence of natural numbers not greater than n. Nonsymmetric coefficients generated in the course of the calculations must be symmetrized since the coefficients in question are subject to conditions (3.4). We replace the summation indices in each addend of the sum over μ in the second term in the left-hand side of (3.6) in the following manner: $j_1, \ldots, j_{\mu-1}, j_{\mu+1}, \ldots, j_\varkappa$ by $i_1, \ldots, i_{\varkappa-1}$, respectively; j_μ by i, and $j_1^\mu, \ldots, j^\mu_{k-\varkappa+1}$ by $i_\varkappa, i_{\varkappa+1}, \ldots, i_k$, respectively. It becomes obvious that all the addends of the sum over μ are identical, so it can be expressed as one addend times $\varkappa$. For a symmetrization of the addend, we consider all combinations $p_1, \ldots, p_{\varkappa-1}$ of natural numbers from $1, \ldots, k$ taken $\varkappa - 1$ at a time (the number of combinations is denoted by $C(k, \varkappa-1)$). Finally, we denote the summation indices $i_{p_1}, \ldots, i_{p_{\varkappa-1}}$ by $j_{p_1}, \ldots, j_{p_{\varkappa-1}}$, and the remaining indices $i_1, \ldots, i_k$ by $j'_\varkappa, j'_{\varkappa+1}, \ldots, j'_k$.

Thus we have completed the transformations

$$\sum_{\mu=1}^{\varkappa}\sum_{j_1,\ldots,j_\varkappa;\,j_1^\mu,\ldots,j^\mu_{k-\varkappa+1}} \alpha^{\nu}_{j_1\ldots j_\varkappa}\varphi^{j_\mu}_{j_1^\mu\ldots j^\mu_{k-\varkappa+1}} y_{j_1}\ldots y_{j_{\mu-1}}$$
$$\times y_{j_{\mu+1}}\ldots y_{j_\varkappa} y_{j_1^\mu}\ldots y_{j^\mu_{k-\varkappa+1}}$$
$$=\varkappa\sum_{i_1,\ldots,i_k;\,i}\alpha^{\nu}_{i_1\ldots i_{\varkappa-1}i}\varphi^{i}_{i_\varkappa\ldots i_k} y_{i_1}\ldots y_{i_k}$$
$$=\varkappa\sum_{i=1}^{n}\sum_{j_1,\ldots,j_k}\frac{1}{C(k,\varkappa-1)} S^{p_1,\ldots,p_{\varkappa-1}}_{1,\ldots,k}\alpha^{\nu}_{j_{p_1}\ldots j_{p_{\varkappa-1}}i}\varphi^{i}_{j'_\varkappa\ldots j'_k} y_{j_1}\ldots y_{j_k} \qquad (3.7)$$
$$(\nu = 1,\ \ldots,\ n).$$

Here $S^{p_1, \ldots, p_{\varkappa-1}}_{1, \ldots, k}$ denotes summation over all combinations $p_1, \ldots$ $\ldots, p_{\varkappa-1}$ of natural numbers from $1, \ldots, k$ taken $\varkappa - 1$ at a time. Note that the numbers $j_{p_1}, \ldots, j_{p_{\varkappa-1}}$ (and possibly all j_p) can be identical since all of them (as well as $j'_\varkappa, j'_{\varkappa+1}, \ldots, j'_k$) independently cover the same set $1, \ldots, n$ in the course of summation. As for the subscripts of i or j, all of them are distinct; this is why combinations are relevant in this analysis.

We now take up the transformation of the second term in the right-hand side of (3.6). We replace the summation indices $j_1^1, \ldots$ $\ldots, j_{\mu_1}^1, \ldots, j_1^\varkappa, \ldots, j_{\mu_\varkappa}^\varkappa$ $(\mu_1 + \ldots + \mu_\varkappa = k)$ by $j_1, \ldots, j_k$. In order to symmetrize the coefficient of $y_{j_1} \ldots y_{j_k}$, we consider all combinations $p_1, \ldots, p_{\mu_1}$ of natural numbers from $1, \ldots, k$ taken μ_1 at a time (the number of combinations is denoted by $C(k, \mu_1)$), then all combinations $p_{\mu_1+1}, \ldots, p_{\mu_1+\mu_2}$ of natural numbers of the remaining $k - \mu_1$ natural numbers from $1, \ldots, k \setminus p_1, \ldots, p_{\mu_1}$ taken μ_2 at a time (the number of combinations is denoted by $C(k-\mu_1, \mu_2)$), and so on, until we come to the combinations $p_{k-\mu_\varkappa-\mu_{\varkappa-1}+1}, \ldots, p_{k-\mu_\varkappa}$ of natural numbers of the remaining $\mu_{\varkappa-1} + \mu_\varkappa$ natural numbers from $1, \ldots, k \setminus$ $\setminus p_1, \ldots, p_{\mu_1}, p_{\mu_1+1}, \ldots, p_{k-\mu_\varkappa-\mu_{\varkappa-1}}$ taken $\mu_{\varkappa-1}$ at a time (the number of combinations is denoted by $C(\mu_{\varkappa-1} + \mu_\varkappa, \mu_{\varkappa-1})$).

Thus we have completed the transformations

$$\sum_{j_1^1, \ldots, j_{\mu_\varkappa}^\varkappa} \alpha^{i_1}_{j_1^1 \ldots j_{\mu_1}^1} \ldots \alpha^{i_\varkappa}_{j_1^\varkappa \ldots j_{\mu_\varkappa}^\varkappa} y_{j_1^1} \ldots y_{j_{\mu_1}^1} \ldots y_{j_1^\varkappa} \ldots y_{j_{\mu_\varkappa}^\varkappa}$$

$$= \sum_{j_1, \ldots, j_k} \frac{1}{C(k, \mu_1)\, C(k-\mu_1, \mu_2) \ldots C(\mu_{\varkappa-1} + \mu_\varkappa, \mu_{\varkappa-1})}$$

$$\times S^{p_{k-\mu_\varkappa-\mu_{\varkappa-1}+1}, \ldots, p_{k-\mu_\varkappa}}_{1, \ldots, k \setminus p_1, \ldots, p_{k-\mu_\varkappa-\mu_{\varkappa-1}}} \ldots S^{p_{\mu_1+1}, \ldots, p_{\mu_1+\mu_2}}_{1, \ldots, k \setminus p_1, \ldots, p_{\mu_1}}$$

$$\times S^{p_1, \ldots, p_{\mu_1}}_{1, \ldots, k} \alpha^{i_1}_{j_{p_1} \ldots j_{p_{\mu_1}}} \alpha^{i_2}_{j_{p_{\mu_1+1}} \ldots j_{p_{\mu_1+\mu_2}}} \times$$

$$\ldots \times \alpha^{i_{\varkappa-1}}_{j_{p_{k-\mu_\varkappa-\mu_{\varkappa-1}+1}} \ldots j_{p_{k-\mu_\varkappa}}} \alpha^{i_\varkappa}_{j_{p_{k-\mu_\varkappa+1}} \ldots j_{p_k}} y_{j_1} \ldots y_{j_k}. \tag{3.8}$$

Here $S^{p_1, \ldots, p_{\mu_1}}_{1, \ldots, k}$ stands for summation over all combinations $p_1, \ldots, p_{\mu_1}$ of natural numbers from $1, \ldots, k$ taken μ_1 at a time; $S^{p_{\mu_1+1}, \ldots, p_{\mu_1+\mu_2}}_{1, \ldots, k \setminus p_1, \ldots, p_{\mu_1}}$ denotes summation over all combinations $p_{\mu_1+1}, \ldots, p_{\mu_1+\mu_2}$ of natural numbers of the remaining $k - \mu_1$ natural numbers from $1, \ldots, k \setminus p_1, \ldots, p_{\mu_1}$ taken μ_2 at a time, and so on.

The symmetrized form of (3.6) is now obtained by using (3.7) and (3.8)

$$\sum \varphi^{\nu}_{j_1 \ldots j_k} y_{j_1} \ldots y_{j_k} + \sum_{\varkappa=2}^{k-1} \frac{\varkappa}{C(k, \varkappa-1)} S^{p_1, \ldots, p_{\varkappa-1}}_{1, \ldots, k}$$

$$\times \sum_{i=1}^{n} \sum_{j_1, \ldots, j_k=1}^{n} \alpha^{\nu}_{j_{p_1} \ldots j_{p_{\varkappa-1}} i} \varphi^{i}_{j'_{\varkappa} \ldots j'_k} y_{j_1} \ldots y_{j_k}$$

$$+ \sum (\lambda_{j_1} + \ldots + \lambda_{j_k} - \lambda_{\nu}) \alpha^{\nu}_{j_1 \ldots j_k} y_{j_1} \ldots y_{j_k}$$

$$= \sum a^{\nu}_{j_1 \ldots j_k} y_{j_1} \ldots y_{j_k} + \sum_{\varkappa=2}^{k-1} \sum_{i_1, \ldots, i_{\varkappa}=1}^{n} a^{\nu}_{i_1 \ldots i_{\varkappa}}$$

$$\times \sum_{\mu_1 + \ldots + \mu_{\varkappa} = k} \sum_{j_1, \ldots, j_k=1}^{n} \frac{1}{C(k, \mu_1) C(k-\mu_1, \mu_2) \ldots C(\mu_{\varkappa-1}+\mu_{\varkappa}, \mu_{\varkappa-1})}$$

$$\times S^{p_{k-\mu_{\varkappa}-\mu_{\varkappa-1}+1}, \ldots, p_{k-\mu_{\varkappa}}}_{1, \ldots, k \setminus p_1, \ldots, p_{k-\mu_{\varkappa}-\mu_{\varkappa-1}}} \ldots S^{p_{\mu_1+1}, \ldots, p_{\mu_1+\mu_2}}_{1, \ldots, k \setminus p_1, \ldots, p_{\mu_1}}$$

$$\times S^{p_1, \ldots, p_{\mu_1}}_{1, \ldots, k} \alpha^{i_1}_{j_{p_1} \ldots j_{p_{\mu_1}}} \alpha^{i_2}_{j_{p_{\mu_1+1}} \ldots j_{p_{\mu_1+\mu_2}}} \times$$

$$\ldots \times \alpha^{i_{\varkappa-1}}_{j_{p_{k-\mu_{\varkappa}-\mu_{\varkappa-1}+1}} \ldots j_{p_{k-\mu_{\varkappa}}}} \alpha^{i_{\varkappa}}_{j_{p_{k-\mu_{\varkappa}+1}} \ldots j_{p_k}} y_{j_1} \ldots y_{j_k} \tag{3.9}$$

$$(\nu = 1, \ldots, n).$$

3.4. Computational alternative in general case. We introduce the symbol

$$\Delta^{\nu}_{j_1 \ldots j_k} = \begin{cases} 1 & \text{if} \quad \lambda_{\nu} = \lambda_{j_1} + \ldots + \lambda_{j_k}, \\ 0 & \text{if} \quad \lambda_{\nu} \neq \lambda_{j_1} + \ldots + \lambda_{j_k} \end{cases} \tag{4.1}$$

$$(\nu, j_1, \ldots, j_k = 1, \ldots, n).$$

The following *alternative* is valid:

(1) Suppose that ν, $j_1, \ldots, j_k$ (and the real parameters of the initial oscillatory system on which λ_{ν}, $\lambda_{j_1}, \ldots, \lambda_{j_k}$ depend) are such that the expression in parentheses in the last sum in each left-hand side of identities (3.9) is distinct from zero, that is, $\Delta^{\nu}_{j_1 \ldots j_k} = 0$. A comparison of the terms containing $y_{j_1} \ldots y_{j_k}$ in the left- and right-hand sides of identities (3.9) shows that the corresponding term in the first sum in the left-hand side definitely vanishes. Indeed,

by using representation (1.3) we write the term containing $y_{j_1} \ldots y_{j_k}$ as

$$y_\nu \cdot \varphi^\nu_{j_1 \ldots j_k} y_{j_1} \cdots y_{j_k} y_\nu^{-1}.$$

For this term, $(\Lambda, Q) = \lambda_{j_1} \cdot 1 + \ldots + \lambda_{j_k} \cdot 1 + \lambda_\nu \cdot (-1) \neq 0$, so that according to (1.3) only those terms remain in the first sum in the left-hand side of (3.9) for which $(\Lambda, Q) = 0$.

Equating coefficients of $y_{j_1} \ldots y_{j_k}$ in (3.9), we obtain the expression for the coefficients of the normalizing transformation (3.2)

$$\alpha^\nu_{j_1 \ldots j_k} = \frac{1 - \Delta^\nu_{j_1 \ldots j_k}}{\lambda_{j_1} + \ldots + \lambda_{j_k} - \lambda_\nu} B^\nu_{j_1 \ldots j_k} \tag{4.2}$$

$$(\lambda_{j_1} + \ldots + \lambda_{j_k} - \lambda_\nu \neq 0;\ \nu,\ j_1, \ldots, j_k = 1, \ldots, n),$$

where

$$B^\nu_{j_1 \ldots j_k} = a^\nu_{j_1 \ldots j_k} + \sum_{\varkappa=2}^{k-1} \Bigg(\sum_{i_1, \ldots, i_\varkappa = 1}^{n} a^\nu_{i_1 \ldots i_\varkappa}$$

$$\times \sum_{\mu_1 + \ldots + \mu_\varkappa = k} \frac{1}{C(k, \mu_1)\, C(k - \mu_1, \mu_2) \ldots C(\mu_{\varkappa-1} + \mu_\varkappa, \mu_{\varkappa-1})}$$

$$\times S^{p_{k-\mu_\varkappa-\mu_{\varkappa-1}+1}, \ldots, p_{k-\mu_\varkappa}}_{1, \ldots, k \setminus p_1, \ldots, p_{k-\mu_\varkappa-\mu_{\varkappa-1}}} \ldots S^{p_{\mu_1+1}, \ldots, p_{\mu_1+\mu_2}}_{1, \ldots, k \setminus p_1, \ldots, p_{\mu_1}}$$

$$\times S^{p_1, \ldots, p_{\mu_1}}_{1, \ldots, k} \alpha^{i_1}_{j_{p_1} \ldots j_{p_{\mu_1}}} \alpha^{i_2}_{j_{p_{\mu_1+1}} \ldots j_{p_{\mu_1+\mu_2}}} \ldots \alpha^{i_{\varkappa-1}}_{j_{p_{k-\mu_\varkappa-\mu_{\varkappa-1}+1}} \ldots j_{p_{k-\mu_\varkappa}}}$$

$$\times \alpha^{i_\varkappa}_{j_{p_{k-\mu_\varkappa+1}} \ldots j_{p_k}} - \frac{\varkappa}{C(k, \varkappa - 1)} S^{p_1, \ldots, p_{\varkappa-1}}_{1, \ldots, k}$$

$$\times \sum_{i=1}^{n} \alpha^\nu_{j_{p_1} \ldots j_{p_{\varkappa-1}}} i \varphi^i_{j'_\varkappa j'_{\varkappa+1} \ldots j'_k} \Bigg) \tag{4.3}$$

$$(\nu,\ j_1, \ldots, j_k = 1, \ldots, n).$$

(2) Suppose that $\nu, j_1, \ldots, j_k$ (and the real parameters of the initial oscillatory system on which $\lambda_\nu, \lambda_{j_1}, \ldots, \lambda_{j_k}$ depend) are such that the expression in parentheses in the last sum in the left-hand side of the identities is zero, that is, $\Delta^\nu_{j_1 \ldots j_k} = 1$. This means, first, that $\alpha^\nu_{j_1 \ldots j_k}$ can be chosen arbitrarily, for example, set to zero, or can be determined by continuity from the values of the real parameters. Second, a comparison of the terms containing $y_{j_1} \ldots y_{j_k}$

in the left- and right-hand sides of identities (3.8) yields the expression for the symmetrized coefficients of the normal form

$$\varphi^{\nu}_{j_1\ldots j_k}=\Delta^{\nu}_{j_1\ldots j_k}B^{\nu}_{j_1\ldots j_k}\quad(\nu,\ j_1,\ \ldots,\ j_k=1,\ \ldots,\ n).\qquad(4.4)$$

Summary. In both formulas, $\Delta^{\nu}_{j_1\ldots j_k}$ plays the part of a 'guard". Indeed, (4.4) yields that $\Delta^{\nu}_{j_1\ldots j_k}=0$ results in $\varphi^{\nu}_{j_1\ldots j_k}=0$ (case (1)). And if $\Delta^{\nu}_{j_1\ldots j_k}=1$ $(\lambda_{j_1}+\ldots+\lambda_{j_k}-\lambda_\nu=0)$, then the quotient in (4.2) becomes meaningless (indefinite); we wish to remind the reader that in this case $\alpha^{\nu}_{j_1\ldots j_k}$ may be assigned an arbitrary value.

Let us clarify the notation used in (4.3) once again. The quantities $a^{\nu}_{j_1\ldots j_k}$ (and $a^{\nu}_{i_1\ldots i_\varkappa}$), $\alpha^{i}_{j_p\ldots j_q}$, and $\varphi^{i}_{j'_\varkappa\ldots j'_k}$ are the symmetrized coefficients of the initial diagonal system (3.1), the normalizing transformation (3.2), and the normal form (3.3), and in addition $\alpha^{j}_{h}=\delta_{jh}$ and $\varphi^{j}_{h}=\lambda_j\delta_{jh}$ (δ_{jh} is the Kronecker delta). Formulas (4.2) and (4.4) are recurrent, and the quantities α and φ in them are coefficients of powers up to $k-1$ inclusive. The numbers $\mu_1,\ \ldots,\ \mu_{k-1}$ are natural numbers; $C(m,\ l)$ is the number of combinations of m elements taken l at a time; $S^{p_1,\ \ldots,\ p_{\mu_1}}_{1,\ \ldots,\ k}$ stands for summation over all combinations $p_1,\ \ldots,\ p_{\mu_1}$ of natural numbers from $1,\ \ldots,\ k$ taken μ_1 at a time; $S^{p_{\mu_1+1},\ \ldots,\ p_{\mu_1+\mu_2}}_{1,\ \ldots,\ k\setminus p_1,\ \ldots,\ p_{\mu_1}}$ denotes summation over all combinations $p_{\mu_1+1},\ \ldots,\ p_{\mu_1+\mu_2}$ of the remaining $k-\mu_1$ natural numbers from $1,\ \ldots,\ k\setminus p_1,\ \ldots,\ p_{\mu_1}$ taken μ_2 at a time, and so on. Finally, $j_\varkappa,\ j'_{\varkappa+1},\ \ldots,\ j'_k$ denote the subscripts that remain in the set $j_1,\ \ldots,\ j_k$ after removing the subset $j_{p_1},\ \ldots,\ j_{p_{\varkappa-1}}$.

3.5. Remark on the transition from symmetrized coefficients to ordinary ones. Let the subscripts $j_1,\ \ldots,\ j_k$ (which, in general, take on values from $1,\ \ldots,\ n$ independently of each other) be distributed in such a manner that the first χ of them $(1\leqslant\chi\leqslant k)$ are distinct, and let j_1 be so distributed that j_1 is found m_{j_1} times, ..., j_χ is found m_{j_χ} times $(m_{j_1}+\ldots+m_{j_\chi}=k)$. The number N of distinct permutations of these subscripts is

$$N=\frac{k!}{m_{j_1}!\ldots m_{j_\chi}!}.\qquad(5.1)$$

This means that the sum

$$\sum_{j_1,\ \ldots,\ j_k=1}^{n}a^{\nu}_{j_1\ldots j_k}x_{j_1}\ldots x_{j_k}$$

includes N similar terms containing $x_{j_1} \ldots x_{j_k}$. Therefore N is the multiplier in the transition from symmetrized to ordinary coefficients, that is, when all the single terms in the sum are distinct

3.6. Formulas for coefficients of fourth-power variables. For $k = 4$, formula (4.3) yields

$$B^{\nu}_{j_1j_2j_3j_4} = a^{\nu}_{j_1j_2j_3j_4} + \frac{1}{2} \sum_{i=1}^{n} (Sa^{\nu}_{j_1i}\alpha^{i}_{j_2j_3j_4} - S\alpha^{\nu}_{j_1i}\varphi^{i}_{j_2j_3j_4} + Sa^{\nu}_{j_1j_2i}\alpha^{i}_{j_3j_4}$$

$$- S\alpha^{\nu}_{j_1j_2i}\varphi^{i}_{j_3j_4}) + \frac{1}{6} \sum_{i,h=1}^{n} Sa^{\nu}_{jh}\alpha^{i}_{j_1j_2}\alpha^{h}_{j_3j_4} \quad (\nu,\ j_1,\ j_2,\ j_3,\ j_4 = 1,\ \ldots,\ n).$$

Here S denotes the sum over all combinations (from numbers 1, 2, 3, 4) of subscripts of j in the first cofactor. For the first two sums this means a circular permutation of the subscripts j_1, j_2, j_3, j_4, and for the remaining sums the subscripts j_1j_2 in the first cofactors are replaced successively by j_1j_3, j_1j_4, j_2j_3, j_2j_4, j_3j_4. For the symmetrized coefficients of the normalizing transformation (1.2) we obtain from (4.1)

$$\alpha^{\nu}_{j_1j_2j_3j_4} = \frac{1 - \Delta^{\nu}_{j_1j_2j_3j_4}}{\lambda_{j_1} + \lambda_{j_2} + \lambda_{j_3} + \lambda_{j_4} - \lambda_{\nu}} B^{\nu}_{j_1j_2j_3j_4} \quad (\nu,\ j_1,\ j_2,\ j_3,\ j_4 = 1,\ \ldots$$
$$\ldots,\ n),$$

where $\Delta^{\nu}_{j_1j_2j_3j_4}$ are defined by (4.1). If $\lambda_{j_1} + \lambda_{j_2} + \lambda_{j_3} + \lambda_{j_4} - \lambda_{\nu} = 0$, the corresponding $\alpha^{\nu}_{j_1j_2j_3j_4}$ may be arbitrary. Finally, (4.4) yields the symmetrized coefficients of the normal form (3.3a)

$$\varphi^{\nu}_{j_1j_2j_3j_4} = \Delta^{\nu}_{j_1j_2j_3j_4} B^{\nu}_{j_1j_2j_3j_4} \quad (\nu,\ j_1,\ j_2,\ j_3,\ j_4 = 1,\ \ldots,\ n).$$

3.7. Case of composite elementary divisors of the matrix of the linear part. Let the linear part of an arbitrary autonomous analytic system of ordinary differential equations be transformed to the Jordan form

$$\frac{dx_{\nu}}{dt} = \lambda_{\nu}x_{\nu} + \delta_{\nu}x_{\nu+1} + \sum_{\varkappa=2}^{\infty} \sum a^{\nu}_{j_1\ldots j_{\varkappa}} x_{j_1} \ldots x_{j_{\varkappa}} \qquad (7.1)$$
$$(\nu = 1,\ \ldots,\ n;\ \delta_n = 0).$$

The vector $\Lambda = (\lambda_1, \ldots, \lambda_n)$ is assumed to have at least one nonvanishing component; $\delta_1, \ldots, \delta_{n-1}$ equal either zero (no composite

elementary divisors) or unity (composite elementary divisors exist). All coefficients of power series (as well as $\lambda_1, \ldots, \lambda_n$) are assumed complex, and the series themselves are assumed convergent in some neighbourhood of the origin. The coefficients of terms that include variables to a power greater than one are assumed symmetrized (see Subsection 3.3). We represent the normalizing transformation in the symmetrized form (3.2). It transforms system (7.1) to the Brjuno normal form ([238j], § 0, II and Ch. I, § 1, I*)

$$\frac{dy_\nu}{dt} = \lambda_\nu y_\nu + \delta_\nu y_{\nu+1} + y_\nu \sum_{(\Lambda,\, Q)=0} g_{\nu Q} y_1^{q_1} \ldots y_n^{q_n} \tag{7.2}$$
$$(\nu = 1, \ldots, n; \quad \delta_n = 0)$$

or, in symmetrized form (the notation is given in Subsection 3.3),

$$\frac{dy_\nu}{dt} = \lambda_\nu y_\nu + \delta_\nu y_{\nu+1} + \sum_{\varkappa=2}^{\infty} \sum \varphi^{\nu}_{j_1 \ldots j_\varkappa} y_{j_1} \ldots y_{j_\varkappa} \tag{7.3}$$
$$(\nu = 1, \ldots, n; \ \delta_n = 0).$$

Similarly to the initial steps traced in Subsection 3.4, substitution of (3.2) into (7.1) yields, by virtue of (7.3),

$$\sum_{\varkappa=2}^{\infty} \sum_{j_1, \ldots, j_\varkappa} \varphi^{\nu}_{j_1 \ldots j_\varkappa} y_{j_1} \ldots y_{j_\varkappa}$$
$$+ \sum_{\varkappa=2}^{\infty} \sum_{j_1, \ldots, j_\varkappa} \sum_{\mu=1}^{\varkappa} \alpha^{\nu}_{j_1 \ldots j_\varkappa} y_{j_1} \ldots y_{j_{\mu-1}} \dot{y}_{j_\mu} y_{j_{\mu+1}} \ldots y_{j_\varkappa}$$
$$= \lambda_\nu \sum_{\varkappa=2}^{\infty} \sum_{j_1, \ldots, j_\varkappa} \alpha^{\nu}_{j_1 \ldots j_\varkappa} y_{j_1} \ldots y_{j_\varkappa} + \delta_\nu$$
$$\times \sum_{\varkappa=2}^{\infty} \sum_{j_1, \ldots, j_\varkappa} \alpha^{\nu+1}_{j_1 \ldots j_\varkappa} y_{j_1} \ldots y_{j_\varkappa}$$
$$+ \sum_{\varkappa=2}^{\infty} \sum_{j_1, \ldots, j_\varkappa} a^{\nu}_{j_1 \ldots j_\varkappa} \prod_{\mu=1}^{\varkappa} \Big(\sum_{\varkappa_\mu=1}^{\infty} \sum_{j_1^\mu, \ldots, j_{\varkappa_\mu}^\mu} \alpha^{j_\mu}_{j_1^\mu \ldots j_{\varkappa_\mu}^\mu} y_{j_1^\mu} \ldots y_{j_{\varkappa_\mu}^\mu} \Big)$$
$$(\nu = 1, \ldots, n; \ \delta_n = 0).$$

---

* In Brjuno's notation, unities in a Jordan elementary matrix are below the principal diagonal, while in our notation they are above it.

We select in these identities the terms with variables to the kth power; in view of (7.3), we obtain, as in (4.1),

$$\sum_{j_1,\ldots,j_k} \varphi^{\nu}_{j_1\ldots j_k} y_{j_1}\cdots y_{j_k} + \sum_{\varkappa=2}^{k-1} \sum_{j_1,\ldots,j_\varkappa} \sum_{\mu=1}^{\varkappa} \alpha^{\nu}_{j_1\ldots j_\varkappa} y_{j_1}\cdots y_{j_{\mu-1}}$$

$$\times y_{j_{\mu+1}}\cdots y_{j_\varkappa} \sum_{j_1^{\mu},\ldots,j^{\mu}_{k-\varkappa+1}} \varphi^{j_\mu}_{j_1^{\mu}\ldots j^{\mu}_{k-\varkappa+1}} y_{j_1^{\mu}}\cdots y_{j^{\mu}_{k-\varkappa+1}}$$

$$+ \sum_{j_1,\ldots,j_k} (\lambda_{j_1} + \ldots + \lambda_{j_k} - \lambda_\nu)\, \alpha^{\nu}_{j_1\ldots j_k} y_{j_1}\cdots y_{j_k}$$

$$+ \sum_{j_1,\ldots,j_k} \alpha^{\nu}_{j_1\ldots j_k} \sum_{\mu=1}^{k} y_{j_1}\cdots y_{j_{\mu-1}} y_{j_{\mu+1}}\cdots y_{j_k} \delta_{j_\mu} y_{j_{\mu+1}}$$

$$= \delta_\nu \sum_{j_1,\ldots,j_k} \alpha^{\nu+1}_{j_1\ldots j_k} y_{j_1}\cdots y_{j_k}$$

$$+ \sum_{j_1,\ldots,j_\varkappa} a^{\nu}_{j_1\ldots j_k} y_{j_1}\cdots y_{j_k} + \sum_{\varkappa=2}^{k-1} \sum_{i_1,\ldots,i_\varkappa} a^{\nu}_{i_1\ldots i_\varkappa}$$

$$\times \sum_{\mu_1+\ldots+\mu_\varkappa=k} \sum_{j_1^1,\ldots,j^{\varkappa}_{\mu_\varkappa}} \alpha^{i_1}_{j_1^1\ldots j^1_{\mu_1}} \cdots \alpha^{i_\varkappa}_{j_1^{\varkappa}\ldots j^{\varkappa}_{\mu_\varkappa}}$$

$$\times y_{j_1^1}\cdots y_{j^1_{\mu_1}}\cdots y_{j_1^{\varkappa}}\cdots y_{j^{\varkappa}_{\mu_\varkappa}} \tag{7.4}$$

$$(\nu = 1, \ldots, n;\ \delta_n = 0),$$

where $\mu_1, \ldots, \mu_{k-1}$ are natural numbers. Symmetrization of non-symmetric coefficients prior to comparing the coefficients of $y_{j_1}\ldots$ $\ldots y_{j_k}$ is carried out in a manner similar to that of Subsection 3.4. As a result, identities (7.4) take on a symmetrized form similar to (3.9)

$$\sum_{j_1,\ldots,j_k=1}^{n} [\varphi^{\nu}_{j_1\ldots j_k} + (\lambda_{j_1} + \ldots + \lambda_{j_k} - \lambda_\nu)\, \alpha^{\nu}_{j_1\ldots j_k}]\, y_{j_1}\cdots y_{j_k}$$

$$+ \sum_{j_1,\ldots,j_k=1}^{n} \alpha^{\nu}_{j_1\ldots j_k} \sum_{\mu=1}^{k} \delta_{j_\mu} y_{j_1}\cdots y_{j_{\mu-1}} y_{j_{\mu+1}}\cdots y_{j_k} y_{j_{\mu+1}}$$

$$-\delta_\nu \sum_{j_1,\ldots,j_k=1}^{\infty} \alpha^{\nu+1}_{j_1\ldots j_k} y_{j_1}\cdots y_{j_k} = \sum_{j_1,\ldots,j_k=1}^{n} B^{\nu}_{j_1\ldots j_k} y_{j_1}\cdots y_{j_k} \tag{7.5}$$

$$(\nu = 1, \ldots, n;\ \delta_n = 0),$$

where $B^{\nu}_{j_1\ldots j_k}$ are given by (4.3). We have already emphasized that all coefficients of interest are symmetrized. From the set of equal coefficients corresponding to distinct permutations of subscripts

we wish to determine the one in which the subscripts are arranged in nondecreasing order. We assume, therefore, that $j_1 \leqslant j_2 \leqslant \ldots \leqslant j_k$ and that all $\alpha^{\nu}_{j'_1 \ldots j'_k}$ $(j'_1 \leqslant j'_2 \leqslant \ldots \leqslant j'_k)$ for which $j'_1 \leqslant j_1, \ldots, j'_k \leqslant j_k$ and $j'_1 + \ldots + j'_k < j_1 + \ldots + j_k$ are determined $(\nu = 1, \ldots, n)$. Equating the coefficients of $y_{j_1} \ldots y_{j_k}$ in identities (7.5), we arrive at a system of n algebraic equations for determining $\alpha^{\nu}_{j_1 \ldots j_k}$ and $\varphi^{\nu}_{j_1 \ldots j_k}$ $(\nu = 1, \ldots, n)$

$$\varphi^1_{j_1 \ldots j_k} + (\lambda_{j_1} + \ldots + \lambda_{j_k} - \lambda_1)\, \alpha^1_{j_1 \ldots j_k}$$
$$+ [\delta_{j_1} \alpha^1_{j_1 - 1 j_2 \ldots j_k} + \cdots + \delta_{j_k} \alpha^1_{j_1 \ldots j_{k-1} j_k - 1}] - \delta_1 \alpha^2_{j_1 \ldots j_k} = B^1_{j_1 \ldots j_k},$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\varphi^{n-1}_{j_1 \ldots j_k} + (\lambda_{j_1} + \ldots + \lambda_{j_k} - \lambda_{n-1})\, \alpha^{n-1}_{j_1 \ldots j_k}$$
$$+ [\delta_{j_1} \alpha^{n-1}_{j_1 - 1 j_2 \ldots j_k} + \ldots + \delta_{j_k} \alpha^{n-1}_{j_1 \ldots j_{k-1} j_k - 1}] - \delta_{n-1} \alpha^n_{j_1 \ldots j_k} = B^{n-1}_{j_1 \ldots j_k},$$
$$\varphi^n_{j_1 \ldots j_k} + (\lambda_{j_1} + \ldots + \lambda_{j_k} - \lambda_n)\, \alpha^n_{j_1 \ldots j_k}$$
$$+ [\delta_{j_1} \alpha^n_{j_1 - 1 j_2 \ldots j_k} + \ldots + \delta_{j_k} \alpha^n_{j_1 \ldots j_{k-1} j_k - 1}] = B^n_{j_1 \ldots j_k}. \qquad (7.6)$$

The quantities

$$\alpha^{\nu}_{j_1 \ldots j_{\mu-1} j_\mu - 1 \ldots j_k} \quad (\mu = 1, \ldots, k;\ \nu = 1, \ldots, n)$$

vanish if one of the subscripts $j_\mu - 1$ does.

System (7.6) is solved stepwise, beginning with the last equation*. The following *computational alternative* holds at each step:

(1) Suppose that $\lambda_{j_1}, \ldots, \lambda_{j_k}$ and n are such that the expression in parentheses in the last equation of (7.6) is nonvanishing. Note that this assumption means $\varphi^n_{j_1 \ldots j_k} = 0$. Indeed, owing to representation (7.3), the corresponding term in identities (7.5) is

$$y_n \varphi^n_{j_1 \ldots j_k} y_{j_1} \ldots y_{j_k} y_n^{-1}.$$

For this term, $(\Lambda, \mathbf{Q}) = \lambda_{j_1} \cdot 1 + \ldots + \lambda_{j_k} \cdot 1 + \lambda_n \cdot (-1) \neq 0$, but (7.2) stipulates that the normal form (7.3) includes only terms with $(\Lambda, \mathbf{Q}) = 0$. The last equation of (7.6) thus gives

$$\alpha^n_{j_1 \ldots j_k} = \frac{1}{\lambda_{j_1} + \ldots + \lambda_{j_k} - \lambda_n} \{B^n_{j_1 \ldots j_k} - [\delta_{j_1} \alpha^n_{j_1 - 1 j_2 \ldots j_k} +$$
$$\ldots + \delta_{j_k} \alpha^n_{j_1 \ldots j_{k-1} j_k - 1}]\}; \quad (\varphi^n_{j_1 \ldots j_k} = 0). \qquad (7.7)$$

---

* This corresponds to beginning with the first equation in Brjuno's notation (see footnote to p. 124).

(2) Suppose that $\lambda_{j_1}, \ldots, \lambda_{j_k}$ and λ_n are such that the expression in parentheses in the last equation of (7.6) is zero. This means, first, that $\alpha^n_{j_1 \ldots j_k}$ can be chosen arbitrarily, for example, set to zero, or can be determined by continuity from the values of the real parameters. Second, the last equation of (3.6) now yields a formula for $\varphi^n_{j_1 \ldots j_k}$

$$\varphi^n_{j_1 \ldots j_k} = B^n_{j_1 \ldots j_k} - [\delta_{j_1} \alpha^n_{j_1 - 1 j_2 \ldots j_k} + \ldots + \delta_{j_k} \alpha^n_{j_1 \ldots j_{k-1} j_k - 1}]. \quad (7.8)$$

In treating the last-but-one equation of (7.6), we have the same alternative

$$\alpha^{n-1}_{j_1 \ldots j_k} = \frac{1}{\lambda_{j_1} + \ldots + \lambda_{j_k} - \lambda_{n-1}} \{B^{n-1}_{j_1 \ldots j_k} + \delta_{n-1} \alpha^n_{j_1 \ldots j_k}$$
$$- [\delta_{j_1} \alpha^{n-1}_{j_1 - 1 j_2 \ldots j_k} + \ldots + \delta_{j_k} \alpha^{n-1}_{j_1 \ldots j_{k-1} j_k - 1}]\} \quad (\varphi^{n-1}_{j_1 \ldots j_k} = 0) \quad (7.9)$$

if $\lambda_{j_1} + \ldots + \lambda_{j_k} - \lambda_{n-1} \neq 0$ and

$$\varphi^{n-1}_{j_1 \ldots j_k} = B^{n-1}_{j_1 \ldots j_k} + \delta_{n-1} \alpha^n_{j_1 \ldots j_k}$$
$$- [\delta_{j_1} \alpha^{n-1}_{j_1 - 1 j_2 \ldots j_k} + \ldots + \delta_{j_k} \alpha^{n-1}_{j_1 \ldots j_{k-1} j_k - 1}] \quad (\alpha^{n-1}_{j_1 \ldots j_k} \text{ is arbitrary}) \quad (7.10)$$

if $\lambda_{j_1} + \ldots + \lambda_{j_k} - \lambda_{n-1} = 0$.

Solutions of the subsequent equations of (7.6) from the $(n - 2)$nd to the first are derived from (7.9) and (7.10) by successive substitution of $n - 1$, $n - 2$, . . ., 2 for n.

CHAPTER VI

NORMAL FORMS OF ARBITRARY-ORDER SYSTEMS IN THE CASE OF ASYMPTOTIC STABILITY IN LINEAR APPROXIMATION

We begin by specifying a class of problems for which the Poincaré theorem (V, 1.3) yields the simplest expression of the normal form, and a statement of the Cauchy problem in general form can be realized efficiently at each step of the approximation. This covers damped oscillatory systems (asymptotically stable in linear approximation) with analytic nonlinearities of a general type. The results are illustrated in Section 2 for mechanical systems with one and two degrees of freedom.

§ 1. Damped Oscillatory Systems

1.1. Reduction to diagonal form. We consider a mechanical system with k degrees of freedom that is described by the differential equations*

$$\ddot{u}_\varkappa + 2\varepsilon_\varkappa \dot{u}_\varkappa + \omega_\varkappa^2 u_\varkappa = f_\varkappa(u_1, \ldots, u_k) + \varphi_\varkappa(\dot{u}_1, \ldots, \dot{u}_k) \qquad (1.1)$$
$$(\varkappa = 1, \ldots, k).$$

Here $f_1, \ldots, f_k$ and $\varphi_1, \ldots, \varphi_k$ are real analytic functions of the corresponding variables in some neighbourhood of zero, and their expansions begin with terms of power two for $f_1, \ldots, f_k$ and not less than two for $\varphi_1, \ldots, \varphi_k$

$$\omega_\varkappa > \varepsilon_\varkappa > 0 \quad (\varkappa = 1, \ldots, k).$$

The case when for some $\varkappa'$ we have $\varepsilon_{\varkappa'} \geqslant \omega_{\varkappa'} > 0$ can be analyzed separately.

We reduce the linear part of system (1.1) to diagonal form by introducing new variables

$$x_j = \frac{i \operatorname{sign} j}{r_{|j|}} (\lambda_{-j} u_{|j|} - \dot{u}_{|j|}) \quad (j = \mp 1, \ldots, \mp k), \qquad (1.2)$$

---

* Instead of $f_\varkappa + \varphi_\varkappa$ in (1.1), we could write $F_\varkappa(u_1, \ldots, u_k, \dot{u}_1, \ldots, \dot{u}_k)$.

where $i=\sqrt{-1}$,

$$\lambda_j = -\varepsilon_{|j|} + i r_{|j|} \operatorname{sign} j \quad (j = \mp 1, \ldots, \mp k),$$
$$r_\varkappa = +\sqrt{\omega_\varkappa^2 - \varepsilon_\varkappa^2} \qquad (\varkappa = 1, \ldots, k).$$

Obviously, $\lambda_{-\varkappa} = \bar{\lambda}_\varkappa$ and $x_{-\varkappa} = \bar{x}_\varkappa$ (negative subscripts are introduced to facilitate further manipulations), and

$$u_\varkappa = \frac{1}{2}(x_{-\varkappa} + x_\varkappa) = \operatorname{Re} x_\varkappa,$$
$$\dot{u}_\varkappa = \frac{1}{2}(\lambda_{-\varkappa}x_{-\varkappa} + \lambda_\varkappa x_\varkappa) = \operatorname{Re}(\lambda_\varkappa x_\varkappa) \tag{1.3}$$
$$(\varkappa = 1, \ldots, k).$$

With these variables, system (1.1) becomes diagonal

$$\dot{x}_j = \lambda_j x_j - \frac{i}{r_{|j|}} \operatorname{sign} j \Big[f_{|j|}\Big(\frac{1}{2}(x_{-1}+x_1), \ldots, \frac{1}{2}(x_{-k}+x_k)\Big)$$
$$+ \varphi_{|j|}\Big(\frac{1}{2}(\lambda_{-1}x_{-1}+\lambda_1 x_1), \ldots, \frac{1}{2}(\lambda_{-k}x_{-k}+\lambda_k x_k)\Big)\Big] \tag{1.4}$$
$$(j = \mp 1, \ldots, \mp k).$$

1.2. Calculation of coefficients of normalizing transformation. Since all $\lambda_{\mp 1}, \ldots, \lambda_{\mp k}$ lie in the left half-plane, condition (2), Chapter V, Subsection 1.3 is satisfied and we have only to consider condition (1) of the Poincaré theorem

$$\lambda_j \neq \sum_{-k}^{k} p_n \lambda_n \quad \Big(j = \mp 1, \ldots, \mp k; \ \sum_{-k}^{k} p_n \geqslant 2\Big) \tag{2.1}$$

for any nonnegative integer p_n. This condition is obviously satisfied for $\varepsilon_1 = \ldots = \varepsilon_k = \varepsilon > 0$; we consider (2.1) satisfied in the case of distinct positive $\varepsilon_1, \ldots, \varepsilon_k$.

Then, by the Poincaré theorem (V, 1.3), there exists a unique reversible normalizing transformation analytic in some neighbourhood of zero

$$x_j = y_j + \sum_{h,\, l=-k}^{k} \alpha^j_{hl} y_h y_l + \sum_{h,\, l,\, m=-k}^{k} \beta^j_{hlm} y_h y_l y_m + \cdots \tag{2.2}$$
$$(j = \mp 1, \ldots, \mp k)$$

that transforms system (1.4) to

$$\dot{y}_j = \lambda_j y_j \quad (j = \mp 1, \ldots, \mp k). \tag{2.3}$$

We limit ourselves to computing the quadratic terms of the normalizing transformation, substituting (2.2) into (1.4)

$$\dot{y}_j+\sum_{h,\,l=-k}^{k}\alpha_{hl}^{j}\,(\dot{y}_h y_l+y_h\dot{y}_l)+\ \dots=\lambda_j y_j+\lambda_j\sum_{h,\,l=-k}^{k}\alpha_{hl}^{j}y_h y_l$$

$$-\frac{i\,\operatorname{sign} j}{r_{|j|}}\Big[f_{|j|}\left(\frac{1}{2}(y_{-1}+y_1),\ \dots,\ \frac{1}{2}(y_{-k}+y_k)\right)$$

$$+\varphi_{|j|}\left(\frac{1}{2}(\lambda_{-1}y_{-1}+\lambda_1 y_1),\ \dots,\ \frac{1}{2}(\lambda_{-k}y_{-k}+\lambda_k y_k)\right)\Big]$$

$$(j=\mp 1,\ \dots,\ \mp k).$$

By (2.3), this yields the identities

$$\sum_{h,l=-k}^{k}(\lambda_h+\lambda_l-\lambda_j)\,\alpha_{hl}^{j}y_h y_l+\ \dots$$

$$=-\frac{1}{2}\frac{i\,\operatorname{sign} j}{r_{|j|}}\sum_{h,l=1}^{k}\Big[\frac{1}{4}\left(\frac{\partial^2 f_{|j|}}{\partial u_h\,\partial u_l}\right)_0(y_{-h}+y_h)(y_{-l}+y_l)$$

$$+\frac{1}{4}\left(\frac{\partial^2\varphi_{|j|}}{\partial\dot{u}_h\,\partial\dot{u}_l}\right)_0(\lambda_{-h}y_{-h}+\lambda_l y_l)(\lambda_{-l}y_{-l}+\lambda_l y_l)\Big]+\ \dots$$

$$(j=\mp 1,\ \dots,\ \mp k),$$

where the zero subscript indicates that all arguments are evaluated at zero, and the terms of power higher than two are ignored. For coefficients of the quadratic terms of expansion (2.2) we obtain

$$\alpha_{hl}^{j}=-\frac{i\,\operatorname{sign} j}{8r_{|j|}}\frac{1}{\lambda_h+\lambda_l-\lambda_j}\left[\left(\frac{\partial^2 f_{|j|}}{\partial u_{|h|}\,\partial u_{|l|}}\right)_0+\lambda_h\lambda_l\left(\frac{\partial^2\varphi_{|j|}}{\partial\dot{u}_{|h|}\,\partial\dot{u}_{|l|}}\right)_0\right]\tag{2.4}$$

$$(j,\ h,\ l=\mp 1,\ \dots,\ \mp k).$$

Obviously, $\alpha_{lh}^{j}=\alpha_{hl}^{j}$ and $\alpha_{-h-l}^{-j}=\overline{\alpha_{hl}^{j}}$ $(j,\ h,\ l=\mp 1,\ \dots,\ \mp k)$, so that we need to write explicitly only one of the four quantities α_{hl}^{j}, α_{lh}^{j}, α_{-h-l}^{-j}, and α_{-l-h}^{-j} (or one pair of the quantities α_{hh}^{j} and α_{-h-h}^{-j}). The denominators $\lambda_h+\lambda_l-\lambda_j$ are distinct from zero by virtue of (2.1). We shall also need the inverse transformation of (2.2); obviously,

$$y_j=x_j-\sum_{h,l=-k}^{k}\alpha_{hl}^{j}x_h x_l+\ \dots\quad (j=\mp 1,\ \dots,\ \mp k)\tag{2.5}$$

$$(y_{-\varkappa}=\bar{y}_{\varkappa},\quad \varkappa=1,\ \dots,\ k).$$

1.3. General solution of the initial system (general solution of the Cauchy problem). Let us find an approximation of the general

solution of system (1.1) that corresponds to the approximation introduced in expansion (2.2). We introduce real variables

$$v_\varkappa = \frac{1}{2}(y_{-\varkappa} + y_\varkappa) = \operatorname{Re} y_\varkappa \quad (\varkappa = 1, \ldots, k). \tag{3.1}$$

By virtue of (2.3), we have

$$\dot{v}_\varkappa = \frac{1}{2}(\lambda_{-\varkappa} y_{-\varkappa} + \lambda_\varkappa y_\varkappa) = \operatorname{Re}(\lambda_\varkappa y_\varkappa) \quad (\varkappa = 1, \ldots, k), \tag{3.2}$$

$$\ddot{v}_\varkappa + 2\varepsilon_\varkappa \dot{v}_\varkappa + \omega_\varkappa^2 v_\varkappa = 0 \qquad (\varkappa = 1, \ldots, k). \tag{3.3}$$

The inverse transformation is

$$y_j = \frac{i \operatorname{sign} j}{r_{|j|}} (\lambda_{-j} v_{|j|} - \dot{v}_{|j|}) \quad (j = \mp 1, \ \ldots, \ \mp k). \tag{3.4}$$

The general solution of (3.3) can be written as

$$\begin{aligned} v_\varkappa &= \frac{1}{r_\varkappa} e^{-\varepsilon_\varkappa t} \left[(r_\varkappa \cos r_\varkappa t + \varepsilon_\varkappa \sin r_\varkappa t)\, v_\varkappa^0 + \sin r_\varkappa t \cdot \dot{v}_\varkappa^0\right], \\ \dot{v}_\varkappa &= \frac{1}{r_\varkappa} e^{-\varepsilon_\varkappa t} \left[-\omega_\varkappa^2 \sin r_\varkappa t \cdot v_\varkappa^0 + (r_\varkappa \cos r_\varkappa t - \varepsilon_\varkappa \sin r_\varkappa t)\, \dot{v}_\varkappa^0\right] \\ &(\varkappa = 1, \ \ldots, \ k), \end{aligned} \tag{3.5}$$

where $v_\varkappa^0 = v_\varkappa(0)$ and $\dot{v}_\varkappa^0 = \dot{v}_\varkappa(0)$. We express $v_\varkappa^0$ and $\dot{v}_\varkappa^0$ in terms of the initial variables $u_\varkappa^0$ and $\dot{u}_\varkappa^0$ by resorting to formulas (3.1), (3.2), (2.5), and (1.2)

$$\begin{aligned} v_\varkappa^0 &= \operatorname{Re}\left(x_\varkappa^0 - \sum_{h,\, l=-k}^{k} \alpha_{hl}^\varkappa x_h^0 x_l^0\right) \\ &= u_\varkappa^0 + \sum_{h,\, l=-k}^{k} \frac{\operatorname{sign}(hl)}{r_{|h|} r_{|l|}} \operatorname{Re}\left[\alpha_{hl}^\varkappa (\lambda_{-h} u_{|h|}^0 - \dot{u}_{|h|}^0)(\lambda_{-l} u_{|l|}^0 - \dot{u}_{|l|}^0)\right], \\ \dot{v}_\varkappa^0 &= \operatorname{Re}\left[\lambda_\varkappa \left(x_\varkappa^0 - \sum_{h,\, l=-k}^{k} \alpha_{hl}^\varkappa x_h^0 x_l^0\right)\right] \\ &= \dot{u}_\varkappa^0 + \sum_{h,\, l=-k}^{k} \frac{\operatorname{sign}(hl)}{r_{|h|} r_{|l|}} \operatorname{Re}\left[\lambda_\varkappa \alpha_{hl}^\varkappa (\lambda_{-h} u_{|h|}^0 - \dot{u}_{|h|}^0)(\lambda_{-l} u_{|l|}^0 - \dot{u}_{|l|}^0)\right] \\ &(\varkappa = 1, \ldots, k). \end{aligned} \tag{3.6}$$

Now we use formulas (1.3), (2.2), and (3.4)

$$u_\varkappa = \operatorname{Re}\left(y_\varkappa + \sum \alpha^\varkappa_{hl} y_h y_l\right)$$

$$= v_\varkappa - \sum_{h,\,l=-k}^{k} \frac{\operatorname{sign}(hl)}{r_{|h|} r_{|l|}} \operatorname{Re}\left[\alpha^\varkappa_{hl} (\lambda_{-h} v_{|h|} - \dot v_{|h|})(\lambda_{-l} v_{|l|} - \dot v_{|l|})\right] \quad (3.7)$$

$$(\varkappa = 1, \ldots, k).$$

Formulas (3.7), (3.5), (3.6), and (2.4) give a solution of the Cauchy problem in general form for the initial system (1.1) with condition (2.1) in the case when expansions (2.2) and (2.5) are restricted to linear and quadratic terms.

§ 2. Examples

2.1. A system with one degree of freedom. For $k = 1$, system (1.1) reduces to the single equation*

$$\ddot u + 2\varepsilon \dot u + \omega^2 u = f(u) + \varphi(\dot u) \quad (\omega > \varepsilon > 0) \qquad (1.1)$$

under the assumptions of Subsection 1.1 concerning the functions f and φ. Formulas (1, 2.4) yield

$$\alpha^1_{-1-1} = \frac{3r + i\varepsilon}{8r(9\omega^2 - 8\varepsilon^2)} [f''(0) + (2\varepsilon^2 - \omega^2 + 2i\varepsilon r)\varphi''(0)],$$

$$\alpha^1_{-11} = \alpha^1_{1-1} = \frac{r + i\varepsilon}{8r\omega^2} [f''(0) + \omega^2 \varphi''(0)],$$

$$\alpha^1_{11} = \frac{-r + i\varepsilon}{8r\omega^2} [f''(0) + (2\varepsilon^2 - \omega^2 - 2i\varepsilon r)\varphi''(0)]$$

$$\left(i = \sqrt{-1},\ r = +\sqrt{\omega^2 - \varepsilon^2}\right).$$

The solution of the Cauchy problem for equation (1.1) is obtained in general form by means of formulas (1, 3.7)

$$u = v - \frac{1}{8(\omega^2 - \varepsilon^2)} \left(\left\{ \frac{1}{9\omega^2 - 8\varepsilon^2} [3f''(0) + (4\varepsilon^2 - 3\omega^2)\varphi''(0)] \right.\right.$$
$$\left. - \frac{1}{\omega^2} [f''(0) - \omega^2 \varphi''(0)] \right\} [(2\varepsilon^2 - \omega^2) v^2 + 2\varepsilon v \dot v + \dot v^2]$$
$$+ 2\varepsilon \left\{ \frac{1}{9\omega^2 - 8\varepsilon^2} [f''(0) + (5\omega^2 - 4\varepsilon^2)\varphi''(0)] \right.$$
$$\left. - \frac{1}{\omega^2} [f''(0) + \omega^2 \varphi''(0)] \right\} v(\varepsilon v + \dot v)$$
$$\left. - \frac{2}{\omega^2} [f''(0) + \omega^2 \varphi''(0)] (\omega^2 v^2 + 2\varepsilon v \dot v + \dot v^2) \right).$$

* Instead of f and φ, we could introduce $F(u, \dot u)$ and treat the case $\varepsilon \geqslant \omega > 0$ separately.

Here (see formulas (1, 3.5))

$$v = \frac{1}{r} e^{-\varepsilon t} [(r \cos rt + \varepsilon \sin rt)\, v(0) + \sin rt \cdot \dot{v}(0)],$$

$$\dot{v} = \frac{1}{r} e^{-\varepsilon t} [-\omega^2 \sin rt \cdot v(0) + (r \cos rt - \varepsilon \sin rt)\, \dot{v}(0)]$$

and, finally, $v(0)$ and $\dot{v}(0)$ are given in terms of the primary initial values by formulas derived from (1, 3.6)

$$\begin{aligned}
v(0) = u(0) + \frac{1}{8r^2} \Bigg(\bigg\{ & \frac{1}{9\omega^2 - 8\varepsilon^2} [3f''(0) - (3\omega^2 - 4\varepsilon^2)\, \varphi''(0)] \\
& - \frac{1}{\omega^2} [f''(0) - \omega^2 \varphi''(0)] \bigg\} [(2\varepsilon^2 - \omega^2)\, u(0)^2 + 2\varepsilon u(0)\, \dot{u}(0) + \dot{u}(0)^2] \\
& + 2\varepsilon \bigg\{ \frac{1}{9\omega^2 - 8\varepsilon^2} [f''(0) + (5\omega^2 - 4\varepsilon^2)\, \varphi''(0)] \\
& - \frac{1}{\omega^2} [f''(0) + \omega^2 \varphi''(0)] \bigg\} u(0)\, [\varepsilon u(0) + \dot{u}(0)] \\
& - \frac{2}{\omega^2} [f''(0) + \omega^2 \varphi''(0)]\, [\omega^2 u(0)^2 + 2\varepsilon u(0)\, \dot{u}(0) + \dot{u}(0)^2] \Bigg),
\end{aligned}$$

$$\begin{aligned}
\dot{v}(0) = \dot{u}(0) + \frac{1}{8r^2} \Bigg(& \frac{1}{9\omega^2 - 8\varepsilon^2} \{ [3f''(0) + (4\varepsilon^2 - 3\omega^2)\, \varphi''(0)] \\
& \times [\varepsilon (3\omega^2 - 4\varepsilon^2)\, u(0)^2 + 2(\omega^2 - 2\varepsilon^2)\, u(0)\, \dot{u}(0) - \varepsilon \dot{u}(0)^2] \\
& - \varepsilon [f''(0) + (5\omega^2 - 4\varepsilon^2)\, \varphi''(0)]\, [(4\varepsilon^2 - \omega^2)\, u(0)^2 \\
& + 4\varepsilon u(0)\, \dot{u}(0) + \dot{u}(0)^2] \} + \frac{4\varepsilon}{\omega^2} [f''(0) + \omega^2 \varphi''(0)] \\
& \times [\omega^2 u(0)^2 + 2\varepsilon u(0)\, \dot{u}(0) + \dot{u}(0)^2] \\
& + 2 \{ f''(0)\, [\varepsilon u(0) + \dot{u}(0)]\, u(0) - \varphi''(0)\, [\omega^2 u(0) + \varepsilon \dot{u}(0)]\, \dot{u}(0) \} \Bigg).
\end{aligned}$$

2.2. Oscillations of a spring-suspended mass with linear damping. Equations (IV, 1, 5.1), with variables changed to

$$\eta \equiv u_1, \quad \xi \equiv u_2,$$

can be expressed in the form

$$\begin{aligned}
& u_1'' + 2\varepsilon u_1' + u_1 = (1 + \gamma + u_1)\, [(1 + \gamma + u_1)^2 + u_2^2]^{-1/2} - 1, \\
& u_2'' + 2\varepsilon u_2' + \frac{\gamma}{1+\gamma} u_2 = u_2\, [(1 + \gamma + u_1)^2 + u_2^2]^{-1/2} - \frac{1}{1+\gamma} u_2,
\end{aligned} \tag{2.1}$$

where

$$\gamma = \frac{\lambda}{l}, \quad \varepsilon = +\frac{b}{2\sqrt{cm}} \quad \left(\varepsilon < \sqrt{\frac{\gamma}{1+\gamma}} \right)$$

are positive dimensionless parameters, and the primes indicate the derivatives with respect to τ. The cases $\sqrt{\gamma/(1+\gamma)} \leqslant \varepsilon \leqslant 1$ and $\varepsilon > 1$ need special analysis.

We calculate the quantities α^j_{hl} by means of (1, 2.4) and list only the nonzero ones (taking into account the remark following formula (1, 2.4))

$$\alpha^1_{-2-2} = \frac{i}{8r_1(1+\gamma)^2}\,\frac{1}{2\lambda_{-2}-\lambda_1},$$

$$\alpha^1_{22} = \frac{i}{8r_1(1+\gamma)^2}\,\frac{1}{2\lambda_2-\lambda_1},$$

$$\alpha^1_{2-2} = \frac{i}{8r_1(1+\gamma)^2}\,\frac{1}{\lambda_2+\lambda_{-2}-\lambda_1},$$

$$\alpha^2_{-1-2} = \frac{i}{8r_2(1+\gamma)^2}\,\frac{1}{\lambda_{-1}+\lambda_{-2}-\lambda_2},$$

$$\alpha^2_{1-2} = \frac{i}{8r_2(1+\gamma)^2}\,\frac{1}{\lambda_1+\lambda_{-2}-\lambda_2},$$

$$\alpha^2_{-12} = \frac{i}{8r_2(1+\gamma)^2}\,\frac{1}{\lambda_{-1}},$$

$$\alpha^2_{12} = \frac{i}{8r_2(1+\gamma)^2}\,\frac{1}{\lambda_1},$$

where

$$\lambda_j = -\varepsilon + ir_j \operatorname{sign} j \quad (j = \mp 1, \quad \mp 2),$$

$$i = \sqrt{-1}, \quad r_1 = +\sqrt{1-\varepsilon^2}, \quad r_2 = +\sqrt{\frac{\gamma}{1+\gamma} - \varepsilon^2}.$$

Formulas (1, 3.7) yield the solution of the Cauchy problem for (2.1) in general form

$$\begin{aligned} u_1 = v_1 - \frac{1}{8r_1r_2^2(1+\gamma)^2}\Bigg(&-\frac{1}{\varepsilon^2+(r_1+2r_2)^2}\{(r_1+2r_2) \\ &\times[(\varepsilon v_2+v_2')^2 - r_2^2v_2^2] + 2\varepsilon r_2v_2(\varepsilon v_2+v_2')\} + 2r_1[(\varepsilon v_2+v_2')^2+r_2^2v_2^2] \\ &+\frac{1}{\varepsilon^2+(r_1-2r_2)^2}\{2\varepsilon r_2v_2(\varepsilon v_2+v_2') - (r_1-2r_2)[(\varepsilon v_2+v_2')^2 - r_2v_2^2]\}\Bigg), \end{aligned}$$

$$\begin{aligned} u_2 = v_2 - \frac{1}{4r_1r_2^2(1+\gamma)^2}\Bigg(&-\frac{1}{\varepsilon^2+(r_1+2r_2)^2}\{(r_1+2r_2) \\ &\times[(\varepsilon v_1+v_1')(\varepsilon v_2+v_2') - r_1r_2v_1v_2] + \varepsilon[r_1v_1(\varepsilon v_2+v_2') + r_2v_2(\varepsilon v_1+v_1')]\} \\ &+2r_1(\varepsilon v_2+v_2')(2\varepsilon v_1+v_1') - \frac{1}{\varepsilon^2+(r_1-2r_2)^2}\{(r_1-2r_2) \\ &\times[r_1r_2v_1v_2 + (\varepsilon v_1+v_1')(\varepsilon v_2+v_2')] + \varepsilon[r_1v_1(\varepsilon v_2+v_2') \\ &- r_2v_2(\varepsilon v_1+v_1')]\}\Bigg). \end{aligned}$$

Here v_1, v_2, v_1', and v_2' are found from (1, 3.5), with τ substituted for t and for $\varepsilon_1 = \varepsilon_2 = \varepsilon$. At the same time, v_1^0, v_2^0, $v_1'^0$, and $v_2'^0$ are given by (1, 3.6), which now have the form

$$
\begin{aligned}
v_1^0 = u_1^0 + \frac{1}{8r_1r_2^2(1+\gamma)^2}\Big(&-\frac{1}{\varepsilon^2+(r_1+2r_2)^2}\{(r_1+2r_2)\,[(\varepsilon u_2^0+u_2'^0)^2 - r_2 u_2^{0^2}] \\
&+ 2\varepsilon r_2 u_2^0(\varepsilon u_2^0+u_2'^0)\} + 2r_1[(\varepsilon u_2^0+u_2'^0)^2 + r_2^2 u_2^{0^2}] \\
&+\frac{1}{\varepsilon^2+(r_1-2r_2)^2}\{2\varepsilon r_2 u_2^0(\varepsilon u_2^0+u_2'^0) - (r_1-2r_2)\,[(\varepsilon u_2^0+u_2'^0)^2 - r_2^2 u_2^{0^2}]\}\Big),
\end{aligned}
$$

$$
\begin{aligned}
v_1'^0 = u_1'^0 + \frac{1}{4r_1r_2^2(1+\gamma)^2}\Big(&\frac{1}{\varepsilon^2+(r_1+2r_2)^2}\{\varepsilon(r_1+r_2)\,[(\varepsilon u_2^0+u_2'^0)^2 - r_2^2 u_2^{0^2}] \\
&- r_2[r_1(r_1+2r_2)-\varepsilon^2]\,u_2^0(\varepsilon u_2^0+u_2'^0)\} - 2\varepsilon r_1[(\varepsilon u_2^0+u_2'^0)^2 + r_2^2 u_2^{0^2}] \\
&+\frac{1}{\varepsilon^2+(r_1-2r_2)^2}\{\varepsilon(r_1-r_2)\,[(\varepsilon u_2^0+u_2'^0)^2 - r_2^2 u_2^{0^2}] \\
&- r_2[r_1(2r_2-r_1)+\varepsilon^2]\,u_2^0(\varepsilon u_2^0+u_2'^0)\}\Big),
\end{aligned}
$$

$$
\begin{aligned}
v_2^0 = u_2^0 + \frac{1}{4r_1r_2^2(1+\gamma)^2}\Big(&-\frac{1}{\varepsilon^2+(r_1+2r_2)^2} \\
&\times\{(r_1+2r_2)\,[(\varepsilon u_1^0+u_1'^0)(\varepsilon u_2^0+u_2'^0) - r_1r_2u_1^0u_2^0] \\
&+\varepsilon\,[r_1u_1^0(\varepsilon u_2^0+u_2'^0) + r_2u_2^0(\varepsilon u_1^0+u_2'^0)]\} \\
&+2r_1(\varepsilon u_2^0+u_2'^0)(2\varepsilon u_1^0+u_1'^0) - \frac{1}{\varepsilon^2+(r_1-2r_2)^2}\{(r_1-2r_2)\,[r_1r_2u_1^0u_2^0 \\
&+(\varepsilon u_1^0+u_1'^0)(\varepsilon u_2^0+u_2'^0)] + \varepsilon\,[r_1u_1^0(\varepsilon u_2^0+u_2'^0) - r_2u_2^0(\varepsilon u_1^0+u_1'^0)]\}\Big),
\end{aligned}
$$

$$
\begin{aligned}
v_2'^0 = u_2'^0 + \frac{1}{4r_1r_2^2(1+\gamma)^2}\Big(&\frac{1}{\varepsilon^2+(r_1+2r_2)^2} \\
&\times\{\varepsilon(r_1+3r_2)\,[(\varepsilon u_1^0+u_1'^0)(\varepsilon u_2^0+u_2'^0) - r_1r_2u_1^0u_2^0] \\
&-[r_2(r_1+2r_2)-\varepsilon^2]\,[r_1u_1^0(\varepsilon u_2^0+u_2'^0) + r_2u_2^0(\varepsilon u_1^0+u_1'^0)]\} \\
&-2r_1\{\varepsilon\,[(\varepsilon u_1^0+u_1'^0)(\varepsilon u_2^0+u_2'^0) + r_2^2u_1^0u_2^0] \\
&+\varepsilon^2u_1^0(\varepsilon u_2^0+u_2'^0) + r_2^2u_2^0(\varepsilon u_1^0+u_1'^0)\} - \frac{1}{\varepsilon^2+(r_1-2r_2)^2} \\
&\times\{\varepsilon(3r_2-r_1)\,[(\varepsilon u_1^0+u_1'^0)(\varepsilon u_2^0+u_2'^0) + r_1r_2u_1^0u_2^0] \\
&+[r_2(r_1-2r_2)+\varepsilon^2]\,[r_2u_2^0(\varepsilon u_1^0+u_1'^0) - r_1u_1^0(\varepsilon u_2^0+u_2'^0)]\}\Big).
\end{aligned}
$$

CHAPTER VII

NORMAL FORMS OF THIRD-ORDER SYSTEMS

The location of the eigenvalues of the matrix representing the linear part of a third-order system in the closed left half-plane of the complex variable λ is of considerable interest in the theory of oscillations. Typical patterns are illustrated in Fig. 10. A case similar to a_1 was discussed in Chapter VI, Section 1. Case a_2 and its

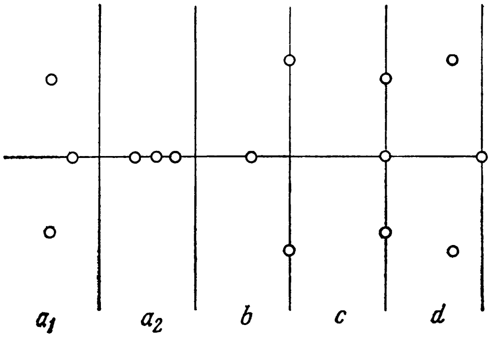

FIG. 10

limit cases when two or three eigenvalues coincide (for example, vanish) would require special analysis. Cases b and c are discussed in Sections 1 and 2; they are of predominant importance. Case d is treated in Section 3. Section 1 concludes with an example from electromechanics. The results of this chapter can also be applied to electromagnetic oscillations of two coupled oscillators when the free oscillations of one of them are described by a nonlinear first-order equation.

§ 1. Case of Two Pure Imaginary Eigenvalues of the Matrix of the Linear Part

1.1. Reduction to normal form. We assume that the initial system is reduced to diagonal form and the independent variable is replaced by $\tau = \omega t$, where $\pm\omega i$ denote pure imaginary eigenvalues of the matrix of the linear part

$$\frac{dx_\nu}{d\tau} = \lambda_\nu x_\nu + \sum a_{jh}^{\nu} x_j x_h + \sum b_{jhk}^{\nu} x_j x_h x_k + \cdots \qquad (1.1)$$

$$(\nu = 0, \ \pm 1).$$

Twice repeated subscripts equal to 0, ± 1 always signify summation; $\lambda_0 = -\delta < 0$, $\lambda_1 = i$, $\lambda_{-1} = -i$. In general, the coefficients a^{ν}_{jh}, $b^{\nu}_{jhk}, \ldots$ are complex and we emphasize that they are symmetrized, that is,

$$a^{\nu}_{hj} = a^{\nu}_{jh}, \quad b^{\nu}_{\{jhk\}} = \text{id.} \quad (\nu,\ j,\ h,\ k = 0,\ \pm 1), \tag{1.2}$$

where $\{jhk\}$ denotes any permutation of j, h, and k.

By the fundamental Brjuno theorem (V, 1.2), there exists a reversible complex change of variables (normalizing transformation)

$$x_j = y_j + \sum \alpha^j_{lm} y_l y_m + \sum \beta^j_{lmn} y_l y_m y_n + \sum \gamma^j_{lmnp} y_l y_m y_n y_p$$
$$+ \sum \delta^j_{lmnpq} y_l y_m y_n y_p y_q + \ldots \quad (j = 0,\ \pm 1), \tag{1.3}$$
$$(\alpha^j_{ml} = \alpha^j_{lm},\ \beta^j_{\{lmn\}} = \text{id.}, \ldots;\quad j,\ l,\ m,\ n,\ p,\ q = 0,\ \pm 1)$$

that reduces system (1.1) to the normal form

$$\frac{dy_\nu}{d\tau} = \lambda_\nu y_\nu + y_\nu \sum_{(\Lambda,\, Q) = 0} g_{\nu Q} y_0^{q_0} y_1^{q_1} y_{-1}^{q_{-1}} \quad (\nu = 0,\ \pm 1). \tag{1.4}$$

Here Λ and Q are vectors with the components λ_0, λ_1, λ_{-1} and q_0, q_1, q_{-1}, respectively; the latter numbers are either zero or integers, and $q_\nu \geqslant -1$, while the remaining q_j are nonnegative and

$$q_0 + q_1 + q_{-1} \geqslant 1. \tag{1.5}$$

Summation in (1.4) is only over resonant terms with exponents satisfying the resonant equation

$$(\Lambda,\ Q) \equiv q_0\lambda_0 + q_1\lambda_1 + q_{-1}\lambda_{-1} = -\delta q_0 + i\,(q_1 - q_{-1}) = 0. \tag{1.6}$$

In this case the solution is obviously

$$q_0 = 0, \quad q_{-1} = q_1. \tag{1.7}$$

Consequently, $q_0 + q_1 + q_{-1} = 2q_1$ in (1.4), and in the case under consideration even-power terms vanish, while for the odd $(2r + 1)$st-power terms we have $q_{-1} = q_1 = r$ $(r = 1, 2, \ldots)$. The normal form (1.4) is therefore given by

$$\frac{dy_1}{d\tau} = iy_1 + y_1 \sum_{r=1}^{\infty} g^1_r y_1^r y_{-1}^r,$$

$$\frac{dy_{-1}}{d\tau} = -iy_{-1} + y_{-1} \sum_{r=1}^{\infty} g^{-1}_r y_1^r y_{-1}^r,$$

$$\frac{dy_0}{d\tau} = -\delta y_0 + y_0 \sum_{r=1}^{\infty} g^0_r y_1^r y_{-1}^r. \tag{1.8}$$

The initial system being real, we have $y_{-1} = \overline{y_1}$ and $g_r^{-1} = \overline{g_r^1}$ ($r = 1, 2, \ldots$); the first equation of (1.8) is a complex conjugate of the second.

1.2. Calculation of coefficients of normalizing transformation and normal form. It was demonstrated in Subsection 1.1 that for the case in question normal forms have no second-power terms. This can also be verified from formulas (V, 3, 2.4), since for all values of the subscripts (V, 3, 2.1) yields $\Delta_{lm}^{\nu} = 0$ (in other words, $\lambda_\nu \neq \lambda_l + \lambda_m$) and $\varphi_{lm}^{\nu} = 0$ ($\nu, l, m = 0, \pm 1$). Consequently, formula (V, 3, 2.2) holds for all values of the subscripts

$$\alpha_{lm}^{\nu} = \frac{a_{lm}^{\nu}}{\lambda_l + \lambda_m - \lambda_\nu} \quad (\nu,\ l,\ m = 0,\ \pm 1). \tag{2.1}$$

Now we wish to calculate the coefficients of third-power terms. Obviously, the subscripts for which $\lambda_\nu = \lambda_l + \lambda_m + \lambda_p$ are $l, m, p = \{\nu, 1, -1\}$, where braces denote, as before, any permutation of the subscripts. In other words,

$$\Delta_{\{\nu 1 -1\}}^{\nu} = 1, \quad \Delta_{lmp}^{\nu} = 0 \tag{2.2}$$
$$(\nu,\ l,\ m,\ p = 0,\ \pm 1; \quad l,\ m,\ p \neq \{\nu,\ 1,\ -1\}).$$

Consequently, $\beta_{\{\nu 1 -1\}}^{\nu}$ can be chosen arbitrarily; we set them to zero

$$\beta_{\{\nu 1 -1\}}^{\nu} = 0 \quad (\nu = 0,\ \pm 1). \tag{2.3}$$

For the remaining values of β_{lmp}^{ν}, formula (V, 3, 2.3) is valid. By (2.2), only the symmetrized coefficients of the normal form $\chi_{\{\nu 1 -1\}}^{\nu}$ are distinct from zero in formulas (V, 3, 2.6). These are obviously related to the coefficients g_1^{ν} of the normal form in representation (1.8)

$$g_1^0 = 6\chi_{0-1-1}^0, \quad g_1^1 = 3\chi_{11-1}^1, \quad g_1^{-1} = 3\chi_{-11-1}^{-1}$$

(the multiplier is equal to the number of distinct permutations of the subscripts of χ). From formulas (V, 3, 2.6) we have

$$g_1^0 = 6b_{01-1}^0 + 4 \sum_{j=0,\ \pm 1} (a_{0j}^0 \alpha_{1-1}^j + a_{1j}^0 \alpha_{-10}^j + a_{-1j}^0 \alpha_{01}^j), \tag{2.4}$$

$$g_1^1 = \overline{g_1^{-1}} = 3b_{11-1}^1 + 2 \sum_{j=0,\ \pm 1} (2a_{1j}^1 \alpha_{1-1}^j + a_{-1j}^1 \alpha_{11}^j). \tag{2.5}$$

It follows from the proof of the fundamental Brjuno theorem (V, 1.2) that transformation (1.3) is nonunique when equation (1.6) has at least one solution Q (nonzero by condition (1.5)). This is discussed in more detail in V, 2.1. Before we discuss the convergence of (1.3), let us consider a specific "branch" of the transformation.

We assume henceforth that if the coefficients of $\gamma^j_{l'm'n'p'}$ and $\delta^j_{l''m''n''p''q''}$ (see (1.3)) and so on vanish in identities (V, 3, 1.6), which are written out to the fourth-, fifth-, etc., powers, then $\gamma^j_{l'm'n'p'} = 0$ and $\delta^j_{l''m''n''p''q''} = 0$, and so on.

This means that in series (V, 2, 1.2) all $h_{\nu Q} = 0$ ($\nu = 0, \pm 1$; $Q \in \mathfrak{M}_\nu$, $(\Lambda, Q) = 0$); that is, series (V, 2, 1.2) are reduced to (V, 2, 1.3) and converge trivially.

We now take up the conditions of convergence of the normalizing transformation (1.3). Equation (1.6) demonstrates that

$$|(\Lambda, Q)| = +\sqrt{q_0^2\delta^2 + (q_1 - q_{-1})^2}.$$

By definition (V, 2, 4.1), we have

$$\omega_k = \Delta = \inf(\delta, 1) \quad (\delta > 0).$$

The left-hand side of condition ω (V, 2.4) becomes

$$-\sum_{k=1}^{\infty} \frac{\ln \Delta}{2^k} = -\ln \Delta, \tag{2.6}$$

and condition ω is obviously satisfied. Note that the case $\delta = 0$ cannot be regarded as a limit case (suffice it to note that (2.6) goes to $+\infty$). The case $\delta = 0$ is analyzed in the next section.

If condition A′ (V, 2.4) is satisfied, then

$$g_r^{-1} \equiv \overline{g_r^1} = -g_r^1 \quad (r = 1, 2, \ldots), \tag{2.7}$$

which is equivalent to the condition that the coefficients g_r^1 ($r = 1, 2, \ldots$) are pure imaginary. This follows from the form of the first and second equations of (1.8). Since in the problem under consideration there is only one pair of conjugate pure imaginary eigenvalues of the matrix of the linear part of the initial system, we in fact have case (1*) of Chapter V, Subsection 2.4, when condition A degenerates to condition A′. Hence, conditions ω and A hold if (2.7) hold, and by the Brjuno theorem (V, 2.4), the normalizing transformation (1.3) with the appropriately calculated coefficients is convergent (provided conditions (2.7) are satisfied in some neighbourhood of zero).

What if not all of $g_1^1, g_2^1, \ldots$ are pure imaginary? In this general case the normalizing transformation is, by Brjuno's hypothesis 1 [238i], smooth (infinitely differentiable).

When conditions (2.7) are satisfied, system (1.8) possesses the first integral

$$y_1 y_{-1} \equiv |y_1|^2 = c_1 \quad (c_1 = |y_1(0)|^2 \geqslant 0), \tag{2.8}$$

obtained by multiplying the first and second equations of (1.8) by $y_{-1} = \overline{y}_1$ and y_1, respectively, and adding them. Substitution of this integral into the third equation of (1.8) yields

$$\frac{dy_0}{d\tau} = (c_2 - \delta)\, y_0 \quad \left(c_2 = \sum_1^{\infty} g_r^0 c_1^r\right). \tag{2.9}$$

The neighbourhood of the origin decomposes into cylinders $c_2 =$ $=$ const, each with an equator (limit cycle). This limit cycle is stable for $c_2 < \delta$ (see (2.9)), although, strictly speaking, it has not been shown that it lies within the domain of convergence of series (2.9), which converges by virtue of the Brjuno theorem (V, 2.4). The limit cycle becomes unstable for $c_2 > \delta$.

1.3. Application of power transformation. The number of linearly independent solutions of equation (1.6), as follows from (1.7), is $d = 1$. Consequently, by the Brjuno theorem (V, 2.3), system (1.8) is integrable in quadratures. The last two equations of (1.8) are independent. The matrix A of the power transformation for these equations is

$$\mathbf{A} = \begin{Vmatrix} 1 & 1 \\ 0 & 1 \end{Vmatrix}$$

([238c], § 2), and the power transformation itself is written as

$$z_1 = y_1 y_{-1} = |\, y_1\,|^2, \quad z_{-1} = y_{-1}$$
$$(y_1 = z_1 z_{-1}^{-1}, \quad y_{-1} = z_{-1}).$$

Applied to the first two equations of (1.8), it yields

$$\frac{d \ln z_1}{d\tau} = \frac{d \ln y_1}{d\tau} + \frac{d \ln y_{-1}}{d\tau} = 2 \sum_{r=1}^{\infty} \operatorname{Re} g_r^1 z_1^r.$$

From the above equation we obtain

$$\frac{1}{2} \int_{|y_1(0)|^2}^{z_1} \frac{dz_1}{z_1 \sum\limits_{r=1}^{\infty} \operatorname{Re} g_r^1 z_1^r} = \tau. \tag{3.1}$$

Inversion of this integral gives

$$z_1 = z_1(\tau).$$

Now in a straightforward manner we derive from the first and third equations of (1.8)

$$y_1(\tau)=y_1(0)\,e^{i\tau}\exp\left[\int_0^\tau \sum_{r=1}^{\infty} g_r^1 z_1(s)^r\,ds\right], \tag{3.2}$$

$$y_0(\tau)=y_0(0)\,e^{-\delta\tau}\exp\left[\int_0^\tau \sum_{r=1}^{\infty} g_r^0 z_1(s)^r\,ds\right]. \tag{3.3}$$

The integration in (3.1)-(3.3) is justified since we chose a smooth normalizing transformation (see Subsection 1.2).

Remark. The same conclusion concerning system (1.8) can be obtained by multiplying the second equation of (1.8) by y_1 and the first by $\bar{y}_1$ and adding them. Our purpose was to illustrate Subsection 2.3 of Chapter V.

Finally, we shall analyze the case when, as in Subsection 1.2, terms of the first, second, and third powers are calculated, that is, when only one coefficient, g_1^1, is known among all g_r^1. Then (3.1) yields

$$\frac{1}{|y_1(0)|^2}-\frac{1}{z_1}=2\,\mathrm{Re}\,g_1^1\tau,$$

whence

$$z_1(\tau)=\frac{1}{|y_1(0)|^{-2}-2\,\mathrm{Re}\,g_1^1\tau}.$$

The constant 2 Re g_1^1 is equivalent to the Lyapunov constant G used for analyzing the critical case of the stability of steady motions with a single pair of pure imaginary roots of the linear approximation ([108a], § 40). Namely, a trivial solution is unstable if Re g_1^1 is positive. If, however,

$$\mathrm{Re}\,g_1^1<0, \tag{3.4}$$

a trivial solution is asymptotically stable.

Within the framework of our approximation (with condition (3.4)) we obtain

$$\int_0^\tau z_1(s)\,ds=\int_0^\tau \frac{ds}{|y_1(0)|^{-2}-2\,\mathrm{Re}\,g_1^1 s}=\ln\,[1-2\,\mathrm{Re}\,g_1^1|y_1(0)|^2\tau]^{-\frac{1}{2\,\mathrm{Re}\,g_1^1}}.$$

Formulas (3.2) and (3.3) now yield

$$y_0(\tau)=y_0(0)\,e^{-\delta\tau}[1-2\,\mathrm{Re}\,g_1^1|y_1(0)|^2\tau]^{-\frac{g_1^0}{2\,\mathrm{Re}\,g_1^1}}, \tag{3.5}$$

$$y_1(\tau)=y_1(0)\,e^{i\tau}[1-2\,\mathrm{Re}\,g_1^1|y_1(0)|^2\tau]^{-\frac{g_1^1}{2\,\mathrm{Re}\,g_1^1}}. \tag{3.6}$$

1.4. Free oscillations of an electric servodrive. A schematic of an electric servodrive is shown in Fig. 11. The controlled element (shaft) is rotated by a servomotor coupled to a reducing gearbox. To be specific, the servomotor is a separately excited dc electric motor. The current in the motor armature is controlled by an amplifier with input voltage V, which is a function of the displacement

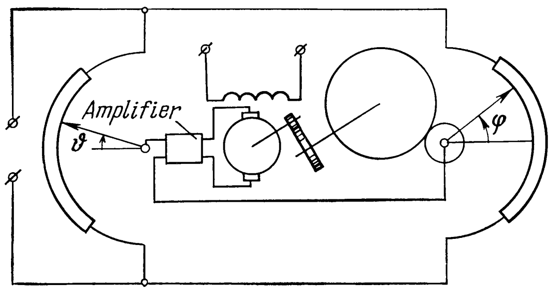

FIG. 11

angle $\vartheta - \varphi$ of the rotors of the control selsyn and the driven selsyn (both selsyns in the schematic are replaced by potentiometers), that is,

$$V = f(\vartheta - \varphi),$$

where φ is the output rotor rotation angle, and ϑ is that of the control rotor with respect to the output rotor rotation angle.

If we neglect the signal delay, the play in each pair of engaged gear wheels, and friction, the equations of motion of the system can be written as

$$(J_1 + J_2)\frac{d^2\varphi}{dt^2} + c\varphi = ki^3,$$

$$L\frac{di}{dt} + Ri + c_1\frac{d\varphi}{dt} = f(\vartheta - \varphi). \qquad (4.1)$$

Here J_1 is the reduced moment of inertia of the servodrive armature and of the rotating parts of the gearbox with respect to the axis of the leading wheel; J_2 is the moment of inertia of the driven rotor with respect to its axis; i is the armature current; L is the self-inductance of the armature circuit; R is the ohmic resistance of the armature circuit; $c_1\,(d\varphi/dt)$ is the counter-emf generated by the rotating armature; ki^3 is the moment of ponderomotive forces reduced to the axis of the leading wheel; and $c\varphi$ is the moment of resistance forces with respect to the axis of the driven rotor.

We assume that the linear dependence of the input voltage (across the armature circuit) on the displacement angle

$$f(\vartheta - \varphi) = v \cdot (\vartheta - \varphi).$$

If we wish to analyze the oscillations of the output rotor (and of the servodrive) around a preset position, we must assume that $\vartheta = 0$. We introduce the dimensionless variables

$$\tau = \sqrt{\frac{c}{J_1 + J_2}}\, t, \quad I = \frac{R}{v}\, i.$$

The equations of motion (4.1) can then be rewritten

$$\frac{dI}{d\tau} = -\delta I - \delta\varphi - \varkappa\varphi', \quad \frac{d\varphi}{d\tau} = \varphi', \quad \frac{d\varphi'}{d\tau} = -\varphi + \gamma I^3, \qquad (4.2)$$

where the following dimensionless positive parameters are used

$$\gamma = \frac{kv^3}{cR^3}, \quad \delta = \frac{R}{L}\sqrt{\frac{J_1 + J_2}{c}}, \quad \varkappa = \frac{c_1 R}{Lv}. \qquad (4.3)$$

If we introduce a vector $\mathbf{u} = (I, \varphi, \varphi')^\tau$, the system becomes

$$\frac{d\mathbf{u}}{d\tau} = \mathbf{C}\mathbf{u} + \begin{pmatrix} 0 \\ 0 \\ \gamma I^3 \end{pmatrix}, \quad \mathbf{C} = \begin{Vmatrix} -\delta & -\delta & -\varkappa \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{Vmatrix}.$$

Obviously, the eigenvalues of the matrix $\mathbf{C}$ are $-\delta$, i, $-i$ ($i = \sqrt{-1}$). We reduce the matrix $\mathbf{C}$ to the Jordan form; this will give $\mathbf{C} = \mathbf{S}\,\mathrm{diag}\,(-\delta, i, -i)\,\mathbf{S}^{-1}$, where

$$\mathbf{S} = \begin{Vmatrix} 1 & -\delta - i\varkappa & -\delta + i\varkappa \\ 0 & \delta + i & \delta - i \\ 0 & -1 + i\delta & -1 - i\delta \end{Vmatrix}.$$

The matrix $\mathbf{S}$ defines the transformation of system (4.2), $\mathbf{u} = \mathbf{S}\mathbf{x}$, $\mathbf{x} = (x_0, x_1, x_{-1})^\tau$ or, in detailed form,

$$\begin{aligned} I &= x_0 - (\delta + i\varkappa)\, x_1 - (\delta - i\varkappa)\, x_{-1} = x_0 - 2\,\mathrm{Re}\,[(\delta + i\varkappa)\, x_1], \\ \varphi &= (\delta + i)\, x_1 + (\delta - i)\, x_{-1} = 2\,\mathrm{Re}\,[(\delta + i)\, x_1], \\ \varphi' &= -(1 - i\delta)\, x_1 - (1 + i\delta)\, x_{-1} = -2\,\mathrm{Re}\,[(1 - i\delta)\, x_1]. \end{aligned} \qquad (4.4)$$

The transformed system becomes

$$\frac{d\mathbf{x}}{d\tau} = \mathrm{diag}\,(-\delta, i, -i)\,\mathbf{x} + \mathbf{S}^{-1}\begin{pmatrix} 0 \\ 0 \\ \gamma I^3 \end{pmatrix}. \qquad (4.5)$$

Calculations result in the following expression for S^{-1}

$$S^{-1}=\frac{1}{2(1+\delta^2)}\begin{Vmatrix} 2(1+\delta^2) & 2(\varkappa+\delta^2) & 2\delta(\varkappa-1) \\ 0 & \delta-i & -1-i\delta \\ 0 & \delta+i & -1+i\delta \end{Vmatrix}.$$

In order to recast system (4.5) in the form of (1.1), it only remains to use the first formula of (4.4) ($a^{\nu}_{lm} = 0$; ν, l, $m = 0, \pm 1$)

$$\frac{dx_\nu}{d\tau}=\lambda_\nu x_\nu+b^{\nu}_{000}x_0^3+b_{111}x_1^3+b^{\nu}_{-1-1-1}x_{-1}^3$$
$$+3b^{\nu}_{001}x_0^2x_1+3b^{\nu}_{00-1}x_0^2x_{-1}+3b^{\nu}_{011}x_0x_1^2+3b^{\nu}_{0-1-1}x_0x_{-1}^2$$
$$+3b^{\nu}_{11-1}x_1^2x_{-1}+3b^{\nu}_{1-1-1}x_1x_{-1}^2+6b^{\nu}_{01-1}x_0x_1x_{-1}\quad(\nu=0,\ \pm 1).$$

Here

$$b^0_{000}=\Delta(\varkappa-1)\delta,$$
$$b^0_{111}=\overline{b^0_{-1-1-1}}=-\Delta(\varkappa-1)\delta(\delta+i\varkappa)^3,$$
$$b^0_{001}=\overline{b^0_{00-1}}=-\Delta(\varkappa-1)\delta(\delta+i\varkappa),$$
$$b^0_{011}=\overline{b^0_{0-1-1}}=\Delta(\varkappa-1)\delta(\delta+i\varkappa)^2,$$
$$b^0_{11-1}=\overline{b^0_{1-1-1}}=-\Delta(\varkappa-1)\delta(\delta^2+\varkappa^2)(\delta+i\varkappa),$$
$$b^0_{01-1}=\Delta(\varkappa-1)\delta(\delta^2+\varkappa^2),$$
$$b^1_{000}=-\frac{1}{2}\Delta(1+i\delta),$$
$$b^1_{111}=\frac{1}{2}\Delta(1+i\delta)(\delta+i\varkappa)^3,$$
$$b^1_{001}=\frac{1}{2}\Delta(1+i\delta)(\delta+i\varkappa),$$
$$b^1_{011}=-\frac{1}{2}\Delta(1+i\delta)(\delta+i\varkappa)^2,$$
$$b^1_{01-1}=-\frac{1}{2}\Delta(\delta^2+\varkappa^2)(1+i\delta),$$
$$b^1_{11-1}=\frac{1}{2}\Delta(\delta^2+\varkappa^2)(1+i\delta)(\delta+i\varkappa),$$
$$b^1_{-1-1-1}=\frac{1}{2}\Delta(1+i\delta)(\delta-i\varkappa)^3,$$
$$b^1_{00-1}=\frac{1}{2}\Delta(1+i\delta)(\delta-i\varkappa),$$
$$b^1_{0-1-1}=-\frac{1}{2}\Delta(1+i\delta)(\delta-i\varkappa)^2,$$
$$b^1_{1-1-1}=\frac{1}{2}\Delta(\delta^2+\varkappa^2)(1+i\delta)(\delta-i\varkappa),\tag{4.6}$$

where $\Delta = \gamma(1+\delta^2)^{-1}$. Since in this problem $a^{\nu}_{lm} = 0$, we obtain from (2.1) $\alpha^{\nu}_{lm} = 0$ (ν, l, $m = 0, \pm 1$), so that formulas (2.4) and (2.5) yield

$$g_1^0 = 6\Delta(\varkappa - 1)\delta(\delta^2 + \varkappa^2),$$
$$g_1^1 = \frac{3}{2}\Delta(\delta^2+\varkappa^2)[(1-\varkappa)\delta + i(\varkappa+\delta^2)].$$

Condition (3.4) of asymptotic stability becomes (see also (4.3))

$$\varkappa > 1 \quad (Lv < c_1 R). \tag{4.7}$$

Assuming that this condition is satisfied, we denote

$$a^2 = -2\operatorname{Re} g_1^1 = 3\Delta(\varkappa-1)\delta(\delta^2+\varkappa^2),$$
$$b^2 = -\frac{1}{2}\frac{\operatorname{Im} g_1^1}{\operatorname{Re} g_1^1} = \frac{1}{2}\frac{\varkappa+\delta^2}{(\varkappa-1)\delta}. \tag{4.8}$$

In this notation, formulas (3.5) and (3.6) become

$$y_0(\tau) = y_0(0)\, e^{-\delta\tau}(1 + a^2 \mid y_1(0) \mid^2 \tau)^2,$$
$$y_1(\tau) = y_1(0)(1 + a^2 \mid y_1(0)\mid^2 \tau)^{-1/2}$$
$$\times \exp i[\tau + b^2 \ln(1 + a^2 \mid y_1(0)\mid^2\tau)]. \tag{4.9}$$

Using formulas (V, 3, 2.3), (2.3), and (4.6), we can rewrite the normalizing transformation (1.3)

$$x_0 = y_0 + \frac{\gamma(\varkappa-1)\delta}{1+\delta^2}\Big\{-\frac{1}{2\delta}y_0^3 - 2\operatorname{Re}\Big[\frac{(\delta+i\varkappa)^3}{\delta+3i}y_1^3\Big]$$
$$+ 6y_0^2 \operatorname{Re}\Big(\frac{\delta+i\varkappa}{\delta-i}y_1\Big) - 3y_0\operatorname{Re}[i(\delta+i\varkappa)^2 y_1^2]$$
$$- 6(\delta^2+\varkappa^2)y_1y_{-1}\operatorname{Re}\Big(\frac{\delta+i\varkappa}{\delta+i}y_1\Big)\Big\} + (4),$$

$$x_1 = y_1 + \frac{1}{2}\frac{\gamma(1+i\delta)}{1+\delta^2}\Big\{\frac{1}{3\delta+i}y_0^3 - \frac{1}{2}i(\delta+i\varkappa)^3 y_1^3$$
$$+\frac{1}{4}i(\delta-i\varkappa)^3 y_{-1}^3 - \frac{3}{2\delta}(\delta+i\varkappa)y_0^2y_1 - \frac{3}{2}\frac{\delta-i\varkappa}{\delta+i}y_0^2y_{-1}$$
$$+3\frac{(\delta+i\varkappa)^2}{\delta-i}y_0y_1^2 + 3\frac{(\delta-i\varkappa)^2}{\delta+3i}y_0y_{-1}^2$$
$$+\frac{3}{2}(\delta^2+\varkappa^2)(\varkappa+i\delta)y_1y_{-1}^2 + 6\frac{\delta^2+\varkappa^2}{\delta+i}y_0y_1y_{-1}\Big\} + (4). \tag{4.10}$$

In order to express the initial values $y_0(0)$ and $y_1(0)$ in terms of $x_0(0)$ and $x_1(0)$, we must invert formulas (4.10); apparently, the inversion is simply the reversal of the sign in front of the braces. It is now possible, by using formulas (4.4) and (4.9), to derive expressions giving the solution of the Cauchy problem in general

form up to the accuracy of third-power terms. We shall not do that here, however, but shall single out the principal part of the solution.

The principal part of a solution is defined, within the approximation made, as the solution of a normal form up to the accuracy of third-power terms that has been transformed by a normalizing transformation up to the accuracy of second-power terms. Formulas (4.4), (4.9), and (4.10) give for the principal part of the rotation angle φ of the output rotor

$$\varphi(\tau) = 2\,\mathrm{Re}\,[(\delta+i)\,x_1(\tau)] \approx 2\,\mathrm{Re}\,[(\delta+i)\,y_1(\tau)]$$

$$\approx 2\,\mathrm{Re}\left\{(\delta+i)\frac{x_1(0)}{\sqrt{1+a^2\,|\,x_1(0)\,|^2\,\tau}}\exp i\,[\tau+b^2\ln(1+a^2\,|\,x_1(0)\,|^2\,\tau)]\right\}$$

$$=\left[1+a^2\frac{\varphi(0)^2+\varphi'(0)^2}{4(1+\delta^2)}\tau\right]^{-1/2}$$

$$\times\left\{\varphi(0)\cos\left[\tau+b^2\ln\left(1+a^2\frac{\varphi(0)^2+\varphi'(0)^2}{4(1+\delta^2)}\tau\right)\right]\right.$$

$$\left.+\varphi'(0)\sin\left[\tau+b^2\ln\left(1+a^2\frac{\varphi(0)^2+\varphi'(0)^2}{4(1+\delta^2)}\tau\right)\right]\right\}. \qquad (4.11)$$

Conclusions. (1) The stability boundary of self-excited vibrations (see (4.7)) is

$$\varkappa = 1 \quad (Lv = c_1R).$$

(2) The actual accuracy of the principal part of the solution is found by comparing (4.11) with the solution of the Cauchy problem. The latter is given by formulas (4.4), (4.9), and (4.10).

§ 2. Case of Neutral Linear Approximation

2.1. Normal form. Let us return to Subsection 1.1 and analyze the case $\delta = 0$, retracing all the steps up to the resonant equation (1, 1.6), which now is written as

$$q_{-1} - q_1 = 0. \qquad (1.1)$$

As before, q_0 and q_1 are "arbitrary", and $q_{-1} = q_1$. Of course (hence the quotation marks) $\mathbf{Q} \in \mathfrak{M}$, that is,

$$q_\nu \geqslant -1, \quad q_j \geqslant 0 \quad (j \neq \nu) \quad (\nu = 0, \pm 1),$$
$$q_0 + q_1 + q_{-1} = q_0 + 2q_1 \geqslant 1.$$

The following set of exponents is possible for the $2r$th-power terms of the normal form ($q_0 + 2q_1 = 2r - 1$; the subscripts on $\mathbf{Q}$ mark the number of the corresponding equation)

$$\mathbf{Q}_0 = (-1,\ r,\ r),\quad (1,\ r-1,\ r-1),\ \ldots,\ (2r-1,\ 0,\ 0),$$
$$\mathbf{Q}_{\pm 1} = \qquad (1,\ r-1,\ r-1),\ \ldots,\ (2r-1,\ 0,\ 0).$$

The values of the vector $\mathbf{Q}$ for the $(2r+1)$st-power terms of the normal form $(q_0+2q_1=2r;\ r=1, 2, \ldots)$ are

$$\mathbf{Q}=(0, r, r),\ \ (2, r-1, r-1), \ldots, (2r, 0, 0),$$

regardless of the number of the equation of the normal form.

When all solutions of the resonant equation (1.1) are found, we represent the normal form (V, 1, 2.4) as

$$\frac{dy_\nu}{d\tau}=y_\nu \sum_{\substack{s\geqslant 0\ (\text{if } \nu\neq 0)\\ s\geqslant -1\ (\text{if } \nu=0)\\ \sigma\geqslant 0}} G^\nu_{s\sigma} y_0^s y_1^\sigma y_{-1}^\sigma \quad (\nu=0, \pm 1), \tag{1.2}$$

or, in detailed form,

$$\frac{dy_\nu}{d\tau}=iy_\nu\delta_{|\nu|1}\operatorname{sign}\nu+y_\nu\sum_{r=1}^{\infty}\sum_{p=0}^{r}[g^\nu_{2(r-p)-1,\,p}y_0^{-1}$$
$$+g^\nu_{2(r-p),\,p}]\,y_0^{2(r-p)}y_1^p y_{-1}^p \quad (\nu=0, \pm 1), \tag{1.2a}$$

where

$$g^{\pm 1}_{-1r}=0 \quad (r=1, 2, \ldots), \tag{1.3}$$

because calculation of $\mathbf{Q}_\nu$ shows that the last two equations $(\nu=\pm 1)$ have no terms containing y_0^{-1}.

Note that the normal form is now more complicated than (1, 1.8). This is not only a complication of notation. First, the number l of eigenvalues of the linear part of the system on the imaginary axis is now equal to the system's order $(l=3)$, so that the Brjuno theorem (V, 2.2) gives no simplifications. Second, all the solutions of the resonant equation (1.1) can be written as

$$\mathbf{Q}=q_0(1, 0, 0)+q_1(0, 1, 1)$$
$$(q_0=-1, 0, 1, 2, \ldots;\quad q_1=0, 1, 2, \ldots),$$

where q_0 and q_1 do not vanish simultaneously. This means that the number d of linearly independent solutions of the resonant equation is two. By the Brjuno theorem of Chapter V, Subsection 2.3, there exists a birational transformation reducing the normal form to system (V, 2, 3.3) (not a normal form), in which the first two equations form a second-order system and the third equation is reducible to quadratures. Actually, this second-order system can be obtained directly by multiplying the second and third equations of (1.2) by $y_{-1}=\overline{y}_1$ and y_1, respectively, and adding them

$$\frac{dy_0}{d\tau}=y_0\sum_{\substack{s\geqslant -1\\ \sigma\geqslant 0}} G^0_{s\sigma}y_0^s z^\sigma,\quad \frac{dz}{d\tau}=2z\sum_{\substack{s,\,\sigma\geqslant 0\\ s+\sigma>0}}\operatorname{Re}G^1_{s\sigma}y_0^s z^\sigma \tag{1.4}$$

($z = y_{-1}y_1 = |\, y_1\,|^2$). Starting with (1.2a), we obtain, in detailed form,

$$\frac{dy_0}{d\tau} = y_0 \sum_{r=1}^{\infty} \sum_{p=0}^{r} [g^0_{2(r-p)-1,p} y_0^{-1} + g^0_{2(r-p),\,p}]\, y_0^{2(r-p)} z^p,$$

$$\frac{dz}{d\tau} = 2z \sum_{r=1}^{\infty} \sum_{p=0}^{r} [\operatorname{Re} g^1_{2(r-p)-1,p} y_0^{-1} + \operatorname{Re} g^1_{2(r-p),p}]\, y_0^{2(r-p)} z^p. \tag{1.4a}$$

2.2. Calculation of coefficients of normalizing transformation and normal form. Let us follow the alternative of Chapter V, Subsection 3.2 and single out all of the subscripts ν, l, and m for which $\lambda_\nu = \lambda_l + \lambda_m$. By virtue of the notation of (V, 3, 2.1),

$$\Delta^0_{00} = \Delta^0_{\{1-1\}} = \Delta^1_{\{01\}} = \Delta^{-1}_{\{0-1\}} = 1, \quad \Delta^\nu_{lm} = 0 \tag{2.1}$$
$$(\nu,\ l,\ m = 0,\ \pm 1;\ \nu,\ l,\ m \neq 0,\ 0,\ 0;\ 0,\ \{1,\ -1\};\ 1,\ \{0,\ 1\};\ -1,\ \{0,\ -1\}).$$

We assume zero values for α^ν_{lm} as defined by case (2) of the alternative

$$\alpha^0_{00} = \alpha^0_{\{1-1\}} = \alpha^1_{\{01\}} = \alpha^{-1}_{\{0-1\}} = 0. \tag{2.2}$$

For the same values of ν, l, and m, formulas (V, 3, 2.4) yield

$$g^0_{10} = \varphi^0_{00} = a^0_{00}, \qquad g^0_{-11} = 2\varphi^0_{1-1} = 2a^0_{1-1},$$
$$g^1_{10} = 2\varphi^1_{01} = 2a^1_{01}, \quad g^{-1}_{10} = 2\varphi^{-1}_{0-1} = 2a^{-1}_{0-1} \tag{2.3}$$

as the quadratic coefficients of the normal form. For the remaining quadratic coefficients of the normalizing transformation we obtain from formula (V, 3, 2.2)

$$\alpha^\nu_{lm} = \frac{a^\nu_{lm}}{\lambda_l + \lambda_m - \lambda_\nu} \tag{2.4}$$
$$(\nu,\ l,\ m = 0,\ \pm 1;\ \nu,\ l,\ m \neq 0,\ 0,\ 0;\ 0,\ \{1,\ -1\};\ 1,\ \{0,\ 1\};\ -1,\ \{0,\ -1\}).$$

In order to find the cubic terms, we first of all single out those subscripts ν, l, m, and p for which $\lambda_\nu = \lambda_l + \lambda_m + \lambda_p$ and assume that

$$\beta^0_{000} = \beta^0_{\{01-1\}} = \beta^1_{\{001\}} = \beta^1_{\{11-1\}} = \beta^{-1}_{\{00-1\}} = \beta^{-1}_{\{1-1-1\}} = 0, \tag{2.5}$$

where, as before, $\{l,\ m,\ p\}$ denotes any permutation of the numbers l, m, and p. Formulas (V, 3, 2.6), together with (2.2), yield the

following expressions for the corresponding cubic coefficients of the normal form

$$
\begin{aligned}
g^0_{20} &= \chi^0_{000} = b^0_{000} + 2\sum_{j=0,\pm1} a^0_{0j}\alpha^j_{00},\\
g^0_{01} &= 6\chi^0_{01-1} = 6b^0_{01-1} + 4\sum_j (a^0_{0j}\alpha^j_{1-1} + a^0_{1j}\alpha^j_{-10} + a^0_{-1j}\alpha^j_{01}),\\
g^1_{20} &= 3\chi^1_{001} = 3b^1_{001} + 2\sum_j (2a^1_{0j}\alpha^j_{01} + a^1_{1j}\alpha^j_{00}),\\
g^1_{01} &= 3\chi^1_{11-1} = 3b^1_{11-1} + 2\sum_j (2a^1_{1j}\alpha^j_{1-1} + a^1_{-1j}\alpha^j_{11}),\\
g^{-1}_{20} &= 3\chi^{-1}_{00-1} = 3b^{-1}_{00-1} + 2\sum_j (2a^{-1}_{0j}\alpha^j_{0-1} + a^{-1}_{-1j}\alpha^j_{00}),\\
g^{-1}_{01} &= 3\chi^{-1}_{1-1-1} = 3b^{-1}_{1-1-1} + 2\sum_j (a^{-1}_{1j}\alpha^j_{-1-1} + 2a^{-1}_{-1j}\alpha^j_{1-1}).
\end{aligned}
\tag{2.6}
$$

From formula (V, 3, 2.3) we obtain the expressions for the remaining cubic terms of the normalizing transformation

$$
\beta^\nu_{lmp} = \frac{1}{\lambda_l + \lambda_m + \lambda_p - \lambda_\nu} \times \left[b^\nu_{lmp} + \frac{2}{3}\sum_{j=0,\pm1} (a^\nu_{lj}\alpha^j_{mp} + a^\nu_{mj}\alpha^j_{pl} + a^\nu_{pj}\alpha^j_{lm})\right] \tag{2.7}
$$

($\nu,\ l,\ m,\ p = 0,\ \pm1;\ \nu,\ l,\ m,\ p \neq 0,\ 0,\ 0,\ 0;\ 0,\ \{0,\ 1,\ -1\};$
$1,\ \{0,\ 0,\ 1\};\ 1,\ \{1,\ 1,\ -1\};\ -1,\ \{0,\ 0,\ -1\};\ -1,\ \{1,\ -1,\ -1\}$).

Taking into account (2.2) and the fact that the initial system is real, the normalizing transformation (1, 1.3) becomes, up to the accuracy of second-power terms,

$$
\begin{aligned}
x_0 &= y_0 + 2\,\mathrm{Re}\,(\alpha^0_{11}y_1^2) + 4\,\mathrm{Re}\,(\alpha^0_{01}y_0y_1) + (3),\\
x_1 = \overline{x}_{-1} &= y_1 + \alpha^1_{00}y_0^2 + \alpha^1_{11}y_1^2 + \alpha^1_{-1-1}y^2_{-1}\\
&\quad + 2\alpha^1_{0-1}y_0y_{-1} + 2\alpha^1_{1-1}\,|\,y_1\,|^2 + (3).
\end{aligned}
\tag{2.8}
$$

The coefficients here are given by formulas (2.4) in terms of the coefficients of the diagonal-type system (1, 1.1) ($\delta = 0$)

$$
\alpha^0_{11} = -\frac{1}{2}ia^0_{11},\quad \alpha^0_{01} = -ia^0_{01},\quad \alpha^1_{00} = ia^1_{00},
$$

$$
\alpha^1_{11} = -ia^1_{11},\quad \alpha^1_{-1-1} = \frac{1}{3}ia^1_{-1-1},\quad \alpha^1_{0-1} = \frac{1}{2}ia^1_{0-1},\quad \alpha^1_{1-1} = ia^1_{1-1}.
$$

The third-power terms of (2.8) can be written by resorting to formulas (2.5) and (2.7).

2.3. Remark on convergence. Following the Brjuno theorem (V, 2.4), we begin with condition ω. Obviously,

$$|(\Lambda, Q)| = |q_1 - q_{-1}|,$$

and by definition (V, 2, 4.1) we obtain

$$\omega_k = 1 \quad (k = 1, 2, \ldots).$$

Condition ω is satisfied (see (V, 2, 4.2)) since its left-hand side is zero.

In the case under discussion, all eigenvalues of the matrix of the linear part of the initial system are on the imaginary axis and are pairwise commensurable; hence, we have case (1*) of Chapter V, Subsection 2.4, when condition A reduces to condition A′

$$\psi_j = \lambda_j y_j a(y_0, y_1, y_{-1}) \quad (j = 0, \pm 1), \tag{3.1}$$

where ψ_j denote the right-hand sides of equations (1.2a). Conditions (3.1) are satisfied only if the right-hand side of the first equation of (1.2a) is identically zero and all coefficients of the right-hand sides of the second equation of (1.2a) are pure imaginary or equal to zero.

With these conditions satisfied, the normalizing transformation in question is convergent in some neighbourhood of zero if all its arbitrarily chosen coefficients are set to zero.

2.4. Conclusions on stability. Taking into account (2.3), we write the terms of (1.2) of up to the second power as

$$\frac{dy_0}{d\tau} = a^0_{00} y_0^2 + 2a^0_{1-1} y_1 \bar{y}_1 + (3),$$

$$\frac{dy_1}{d\tau} = iy_1 + 2a^1_{01} y_0 y_1 + (3),$$

$$\frac{d\bar{y}_1}{d\tau} = -i\bar{y}_1 + 2\bar{a}^1_{01} y_0 \bar{y}_1 + (3). \tag{4.1}$$

Multiplying the second equation by $\bar{y}_1$ and the third by y_1 and adding, we obtain the truncated system of equations

$$\frac{dy_0}{d\tau} = a^0_{00} y_0^2 + 2a^0_{1-1} \rho^2 \qquad \frac{d\rho}{d\tau} = 2 \operatorname{Re} a^1_{01} y_0 \rho \tag{4.2}$$

with real coefficients and a nonnegative variable $\rho = |y_1|$ (in contrast to (1.4a), where $z = \rho^2$). Note that system (4.2) is homogeneous and therefore integrable. This property is typical of all truncated second-order systems because, in the nomenclature ([238c], § 2), they have "dimension 1".

In order to reveal the instability of the trivial solution of system (4.2), it is sufficient (see [40a], Sec. 13) to find a single trajectory outside the fixed domain

$$\tau \geqslant \tau_0, \quad y_0^2 + \rho^2 \leqslant R^2$$

for any arbitrarily small initial perturbations $y_0^0 = y_0(\tau_0)$ and $\rho_0 = \rho(\tau_0)$.

We consider some possible situations.

(1) Suppose that $a_{00}^0 \neq 0$. We consider those solutions of system (4.2) that start at the y_0 axis, that is, those for which $\rho_0 = 0$. The second equation of (4.2) yields $(d\rho/d\tau)_0 = 0$; consequently, for these solutions $\rho(\tau) \equiv 0$. We impose the condition

$$\operatorname{sign} y_0^0 = \operatorname{sign} a_{00}^0$$

on the arbitrarily small initial value y_0^0. As follows from the first equation of (4.2), for these solutions

$$|y_0(\tau)| > |a_{00}^0| \, y_0^{0^2} \tau,$$

which means instability of the trivial solution of system (4.2) for $a_{00}^0 \neq 0$.

(2) Suppose that $a_{00}^0 = 0$, but $a_{1-1}^0 \neq 0$, $\operatorname{Re} a_{01}^1 \neq 0$. Dividing the first equation of (4.2) by the second, we obtain

$$\frac{dy_0}{d\rho} = \frac{a_{1-1}^0}{\operatorname{Re} a_{01}^1} \cdot \frac{\rho}{y_0},$$

which yields the first integral of system (4.2) for $a_{00}^0 = 0$

$$y_0^2 - \frac{a_{1-1}^0}{\operatorname{Re} a_{01}^1} \rho^2 = c.$$

Obviously, the trivial solution of system (4.2) is stable (the origin is a *central point*) if

$$a_{00}^0 = 0, \quad a_{1-1}^0 \operatorname{Re} a_{01}^1 < 0 \tag{4.3}$$

and unstable (the origin is a *saddle point*) if

$$a_{00}^0 = 0, \quad a_{1-1}^0 \operatorname{Re} a_{01}^1 > 0.$$

(3) Suppose that $a_{00}^0 = a_{1-1}^0 = 0$, $\operatorname{Re} a_{01}^1 \neq 0$. The first equation of (4.2) yields $y_0(\tau) \equiv y_0^0$. Choosing y_0^0 on the basis of the condition $\operatorname{sign} y_0^0 = \operatorname{sign} \operatorname{Re} a_{01}^1$, we obtain, by the second equation of (4.2), that for such solutions $\rho \to \infty$ as $\tau \to \infty$ and $\rho_0 > 0$; hence, the trivial solution of system (4.2) is unstable.

(4) Finally, suppose that $a_{00}^0 = \operatorname{Re} a_{01}^1 = 0$, $a_{1-1}^0 \neq 0$. The second equation of (4.2) now gives $\rho(\tau) \equiv \rho_0$. The trajectory for which $\operatorname{sign} y_0^0 = \operatorname{sign} a_{1-1}^0$, $\rho_0 > 0$ tends to infinity; this again indicates instability.

Recapitulating, the trivial solution of system (4.2) (and consequently, of the truncated system (4.1)) is stable only if conditions (4.3) are satisfied. The case $a_{00}^0 = a_{1-1}^0 = \operatorname{Re} a_{01}^1 = 0$ requires that the analysis be expanded to terms of at least third power.

Remark. The critical case of one vanishing and two pure imaginary roots is discussed for steady motions by Kamenkov ([83], vol. I) and Malkin ([111c], § 96). It should be emphasized that system (4.2) has only three coefficients; this fact, together with the condition of nonnegative ρ*, determined the specifics of this investigation in comparison to the general case ([111c], §§ 94, 96). It cannot be stated, however, that a normalizing transformation (and hence, a normal form) can be chosen analytic in some neighbourhood of zero. This consideration precludes the generalizing of conclusions on the stability of the trivial solution of system (4.2) to that of unperturbed motion (of the trivial solution of the initial system (1, 1.1) for $\delta = 0$). Note that a stationary point is unstable for $a_{00}^0 \neq 0$ not only in the truncated but also in the complete system (4.2). This corresponds to hypothesis 2 [238i], which has not yet been proved. This hypothesis must be applied to the solution $\rho \equiv 0$. If, however, conditions (4.3) are satisfied, the truncated system is characterized by a stability of a neutral type that can be replaced by instability as a result of neglected terms of higher powers.

2.5. Integration of normal form in quadratic approximation. Returning to the truncated system (4.2) again, we first consider several special cases, always assuming the arithmetic value of the square root.

(1) Suppose that $\operatorname{Re} a_{01}^1 = 0$, $a_{00}^0 \neq 0$, and $a_{1-1}^0 \neq 0$. The second equation of (4.2) yields $\rho \equiv \rho_0$, and the first becomes

$$\frac{dy_0}{d\tau} = a_{00}^0 y_0^2 + 2a_{1-1}^0 \rho_0^2.$$

After integration, this equation gives

(a) if $a \equiv a_{00}^0/(2a_{1-1}^0) > 0$, then

$$y_0 = \frac{y_0^0 + \dfrac{\rho_0}{\sqrt{a}} \tan \dfrac{a_{00}^0 \rho_0}{\sqrt{a}} \tau}{1 - \dfrac{\sqrt{a}\, y_0^0}{\rho_0} \tan \dfrac{a_{00}^0 \rho_0}{\sqrt{a}} \tau};$$

(b) if $a < 0$, then

$$y_0 = \frac{\rho_0}{\sqrt{-a}} \frac{1 + ke^{c\tau}}{1 - ke^{c\tau}} \quad \left(k = \frac{\sqrt{-a}\, y_0^0 - \rho_0}{\sqrt{-a}\, y_0^0 + \rho_0}, \quad c = \frac{2a_{00}^0 \rho_0}{\sqrt{-a}}\right)$$

(here and below $\tau_0 = 0$, $y_0^0 = y_0(0)$).

---

* Chetaev's theorem on instability can only be applied if there is a domain containing a segment of the y_0 axis (see [111c], p. 414), which is impossible for $\rho \geqslant 0$.

(2) Suppose that $\operatorname{Re} a_{01}^{1} = a_{00}^{0} = 0$ and $a_{1-1}^{0} \neq 0$. The solution of system (4.2) is obvious

$$\rho = \rho_0, \quad y_0 = 2a_{1-1}^{0}\rho_0^2\tau + y_0^0.$$

(3) Suppose that $\operatorname{Re} a_{01}^{1} = a_{1-1}^{0} = 0$ and $a_{00}^{0} \neq 0$. The solution of system (4.2) is again obvious

$$\rho = \rho_0, \quad y_0 = \left(\frac{1}{y_0^0} - a_{00}^{0}\tau\right)^{-1}.$$

(4) Suppose that $a_{00}^{0} = 0$, $a_{1-1}^{0} \neq 0$, and $\operatorname{Re} a_{01}^{1} \neq 0$. Dividing the first equation of (4.2) by the second and then integrating, we obtain the first integral of system (4.2)

$$y_0^2 - y_0^{0^2} = \frac{a_{1-1}^{0}}{\operatorname{Re} a_{01}^{1}} (\rho^2 - \rho_0^2).$$

Substitution of this integral into the first equation of (4.2) yields

$$\frac{dy_0}{d\tau} = 2 \operatorname{Re} a_{01}^{1} (y_0^2 - R) \quad \left(R = y_0^{0^2} - \frac{a_{1-1}^{0}}{\operatorname{Re} a_{01}^{1}} \rho_0^2\right).$$

In a manner similar to (1), we consider the following cases:

(a) if $R < 0$ (this is possible only if $a_{1-1}^{0}/\operatorname{Re} a_{01}^{1} > 0$), then

$$y_0 = \frac{y_0^0 + \sqrt{-R} \tan (2\sqrt{-R} \operatorname{Re} a_{01}^{1}\tau)}{1 - \dfrac{y_0^0}{\sqrt{-R}} \tan (2\sqrt{-R} \operatorname{Re} a_{01}^{1}\tau)},$$

$$\rho^2 = -R \frac{\operatorname{Re} a_{01}^{1}}{a_{1-1}^{0}} \left(1 - \frac{y_0^{0^2}}{R}\right) \frac{\sec^2 (2\sqrt{-R} \operatorname{Re} a_{01}^{1}\tau)}{\left[1 - \dfrac{y_0^0}{\sqrt{-R}} \tan (2\sqrt{-R} \operatorname{Re} a_{01}^{1}\tau)\right]^2}.$$

(b) if $R = 0$ (again, this is possible only if $a_{1-1}^{0}/\operatorname{Re} a_{01}^{1} > 0$), then

$$y_0 = \left(\frac{1}{y_0^0} - 2 \operatorname{Re} a_{01}^{1}\tau\right)^{-1}, \quad \rho^2 = \frac{\operatorname{Re} a_{01}^{1}}{a_{1-1}^{0}} \left(\frac{1}{y_0^0} - 2 \operatorname{Re} a_{01}^{1}\tau\right)^{-2}.$$

(c) if $R > 0$, then

$$y_0 = -\sqrt{R} \frac{(\sqrt{R} - y_0^0)\, e^{H\tau} - (\sqrt{R} + y_0^0)\, e^{-H\tau}}{(\sqrt{R} - y_0^0)\, e^{H\tau} + (\sqrt{R} + y_0^0)\, e^{-H\tau}} \quad (H = 2\sqrt{R} \operatorname{Re} a_{01}^{1}),$$

$$\rho^2 = -4 \frac{\operatorname{Re} a_{01}^{1}}{a_{1-1}^{0}} R (R - y^{0^2}) \left[(\sqrt{R} - y_0^0)\, e^{H\tau} + (\sqrt{R} + y_0^0)\, e^{-H\tau}\right]^{-2}. \tag{5.1}$$

Obviously, $\rho(\infty) = 0$ and $y_0(\infty) = -\sqrt{R}$ for $\operatorname{Re} a_{01}^{1} > 0$ and $y_0(\infty) = \sqrt{R}$ for $\operatorname{Re} a_{01}^{1} < 0$. Each solution of system (4.2) traces an arc of the ellipse (5.1) in the ρy_0 plane along one of the two paths shown in Fig. 12 (the choice of path is determined by sign $\operatorname{Re} a_{01}^{1}$).

Note that whatever the initial values of ρ_0 and y_0^0 $(\rho_0 + y_0^{0^2} > 0)$, case (c) is possible only if (4.3) is satisfied.

(5) Suppose that $a^0_{1-1} = 0$, $a^0_{00} \neq 0$, and $\operatorname{Re} a^1_{01} \neq 0$. In this case integration of system (4.2) yields

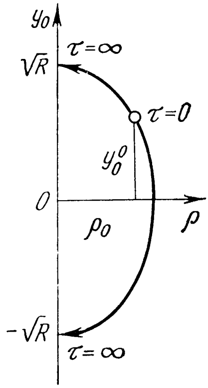

FIG. 12

$$y_0 = \left(\frac{1}{y_0^0} - a^0_{00}\tau\right)^{-1},$$

$$\rho = \rho_0\left(1 - a^0_{00} y^0_0 \tau\right)^{-\frac{2\operatorname{Re} a^1_{01}}{a^0_{00}}}.$$

We consider now the general case (when all three coefficients in system (4.2) are distinct from zero). Dividing the first equation of (4.2) by the second, we obtain

$$\frac{dy_0}{d\rho} = \frac{\alpha}{\gamma}\frac{y_0}{\rho} + \frac{\beta}{\gamma}\frac{\rho}{y_0}$$

$$(\alpha = a^0_{00}, \quad \beta = 2a^0_{1-1}, \quad \gamma = 2\operatorname{Re} a^1_{01}).$$

The substitution $y_0 = \rho u(\rho)$ transforms this equation to

$$\rho u \frac{du}{d\rho} = \left(\frac{\alpha}{\gamma} - 1\right)u^2 + \frac{\beta}{\gamma}. \tag{5.2}$$

(6) Suppose that $a^0_{00} = 2\operatorname{Re} a^1_{01} \neq 0$ $(\alpha = \gamma)$ and $a^0_{1-1} \neq 0$. Integration of equation (5.2) yields

$$y_0 = \pm\rho\sqrt{\frac{y_0^{0^2}}{\rho_0^2} + 2\frac{\beta}{\gamma}\ln\frac{\rho}{\rho_0}}.$$

Substitution of y_0 into the second equation of (4.2) leads to the integral

$$\pm\frac{1}{\gamma}\int\limits_{\rho_0}^{\rho}\frac{d\rho}{\rho^2\sqrt{\frac{y_0^{0^2}}{\rho_0^2} + 2\frac{\beta}{\gamma}\ln\frac{\rho}{\rho_0}}} = \tau.$$

Inversion of this integral yields $\rho = \rho(\tau; \rho_0, y_0^0)$.

(7) Finally, suppose that $a^0_{00} \neq 0$, $a^0_{1-1} \neq 0$, $\operatorname{Re} a^0_{01} \neq 0$, and $a^0_{00} \neq 2\operatorname{Re} a^1_{01}$ $(\alpha \neq \gamma)$. The first integral of system (4.2) is found from (5.2)

$$\left|\frac{(\alpha-\gamma)\,y_0^2 + \beta\rho^2}{(\alpha-\gamma)\,y_0^{0^2} + \beta\rho_0^2}\right| = \left(\frac{\rho}{\rho_0}\right)^{2\frac{\alpha}{\gamma}},$$

whence

$$y_0^2 = \left|\left(y_0^{0^2} + \frac{\beta}{\alpha-\gamma}\rho_0^2\right)\left(\frac{\rho}{\rho_0}\right)^{2\frac{\alpha}{\gamma}} - \frac{\beta}{\alpha-\gamma}\rho^2\right|.$$

The second equation of (4.2) leads to the integral

$$\pm\frac{1}{\gamma}\int\limits_{\rho_0}^{\rho}\rho^{-1}\left|\left(y_0^{0^2}+\frac{\beta}{\alpha-\gamma}\rho_0^2\right)\left(\frac{\rho}{\rho_0}\right)^{2-\frac{\alpha}{\gamma}}-\frac{\beta}{\alpha-\gamma}\rho^2\right|^{-1/2}d\rho=\tau,$$

which gives $\rho=\rho(\tau;\ \rho_0,\ y_0^0)$.

An analysis of cases (1)-(7) shows that only in case (4) are all the solutions of system (4.2) bounded for any $\tau\geqslant 0$ under conditions (4.3). This coincides with the conclusions of Subsection 2.4. The normalizing transformation (2.8) makes it possible to write out the general solution of the initial system (1, 1.1) for $\delta=0$, up to second-power terms. It is also possible to determine conditional stability regions in the space of initial values in cases when the trivial solution of system (4.2) is unstable.

2.6. Example. We consider the equation

$$\dddot{u}+b^2\dot{u}=cu^3\quad(b>0,\quad c\neq 0),$$

or, if dimensionless time $\tau=bt$ is introduced, the equation

$$u'''+u'=\gamma u^3\quad\left({}'=\frac{d}{d\tau},\quad\gamma=\frac{c}{b^3}\right),\tag{6.1}$$

which can be written as the system of equations

$$\frac{d\mathbf{u}}{d\tau}=\mathbf{Au}+\begin{pmatrix}0\\0\\\gamma u^3\end{pmatrix}\quad\left(\mathbf{u}=\begin{pmatrix}u\\u'\\u''\end{pmatrix},\quad\mathbf{A}=\begin{Vmatrix}0&1&0\\0&0&1\\0&-1&0\end{Vmatrix}\right).\tag{6.2}$$

The eigenvalues of the matrix $\mathbf{A}$ are $0,\ i,\ -i$. The matrix $\mathbf{S}$ (and its inverse) that reduces $\mathbf{A}$ to diagonal form is

$$\mathbf{S}=\begin{Vmatrix}1&1&1\\0&i&-i\\0&-1&-1\end{Vmatrix},\quad\mathbf{S}^{-1}=\begin{Vmatrix}1&0&1\\0&-\frac{1}{2}i&-\frac{1}{2}\\0&\frac{1}{2}i&-\frac{1}{2}\end{Vmatrix}.\tag{6.3}$$

The substitution

$$\mathbf{u}=\mathbf{Sx},\quad\mathbf{x}=\begin{pmatrix}x_0\\x_1\\x_{-1}\end{pmatrix}\tag{6.4}$$

reduces system (6.2) to diagonal form

$$\frac{d\mathbf{x}}{d\tau}=\operatorname{diag}(0,\ i,\ -i)+\mathbf{S}^{-1}\begin{pmatrix}0\\0\\\gamma(x_0+x_1+x_{-1})^3\end{pmatrix},$$

or, in detailed form,

$$\frac{dx_0}{d\tau} = \gamma (x_0 + x_1 + x_{-1})^3,$$

$$\frac{dx_1}{d\tau} = ix_1 - \frac{1}{2} \gamma (x_0 + x_1 + x_{-1})^3,$$

$$\frac{dx_{-1}}{d\tau} = -ix_{-1} - \frac{1}{2} \gamma (x_0 + x_1 + x_{-1})^3. \tag{6.5}$$

Since there are no quadratic terms in system (6.5), no such terms will be present in normal form (1.2a), which becomes, within cubic terms,

$$\frac{dy_0}{d\tau} = g_{20}^0 y_0^3 + g_{01}^0 y_0 y_1 y_{-1} + (4),$$

$$\frac{dy_1}{d\tau} = iy_1 + g_{20}^1 y_0^2 y_1 + g_{01}^1 y_1^2 y_{-1} + (4),$$

$$\frac{dy_{-1}}{d\tau} = -iy_{-1} + g_{20}^{-1} y_0^2 y_{-1} + g_{01}^{-1} y_1 y_{-1}^2 + (4). \tag{6.6}$$

The coefficients of the normal form (6.6) are found by means of formulas (2.6), in which the values of b_{lmn}^{ν} are given by (6.5)

$$g_{20}^0 = \gamma, \quad g_{01}^0 = 6\gamma, \quad g_{20}^1 = g_{01}^{-1} = g_{20}^{-1} = g_{01}^{-1} = -\frac{3}{2}\gamma.$$

Multiplying the second equation of (6.6) by $y_{-1} = \bar{y}_1$ and the third by y_1 and then adding them, we obtain the system

$$\frac{dy_0}{d\tau} = \gamma y_0 (y_0^2 + 6\rho^2), \quad \frac{d\rho}{d\tau} = -\frac{3}{2}\gamma\rho (y_0^2 + \rho^2) \tag{6.7}$$

$$(\rho^2 = y_1 y_{-1} = |y_1|^2).$$

System (6.7) is integrable. Integration, however, is not necessary in this particular case, since it is immediately apparent that the trivial solution of (6.7) is unstable for any $\gamma \neq 0$.

§ 3. Case of a Zero Eigenvalue of the Matrix of the Linear Part

3.1. Normal form and normalizing transformation. Let us return again to system (1, 1.1), with t for an independent variable and $\lambda_0 = 0$, $\lambda_1 = -\gamma + i\omega$, and $\lambda_{-1} = -\gamma - i\omega$ ($\gamma > 0$, $\omega > 0$). The resonant equation (1, 1.6) now becomes

$$(\Lambda, Q) = 0 \cdot q_0 - \gamma (q_1 + q_{-1}) + i\omega (q_1 - q_{-1}) = 0, \tag{1.1}$$

and its solutions under condition (1, 1.5) are

$$q_1 = q_{-1} = 0, \quad q_0 = 1, 2, 3, \ldots. \tag{1.2}$$

The normal form (V, 1, 2.4) now takes the form

$$\frac{dy_0}{dt} = y_0 \sum_{s=1}^{\infty} g_s^0 y_0^s,$$

$$\frac{dy_1}{dt} = (-\gamma + i\omega)\, y_1 + y_1 \sum_{s=1}^{\infty} g_s^1 y_0^s,$$

$$\frac{dy_{-1}}{dt} = (-\gamma - i\omega)\, y_{-1} + y_{-1} \sum_{s=1}^{\infty} g_s^{-1} y_0^s, \tag{1.3}$$

where $y_{-1} = \overline{y_1}$ and $g_s^{-1} = \overline{g_s^1}$.

Following the alternative of Chapter V, Subsection 3.2, we single out all the values of the subscripts ν, l, and m for which $\lambda_\nu = \lambda_l + \lambda_m$ and set

$$\alpha_{00}^0 = \alpha_{\{01\}}^1 = \alpha_{\{0-1\}}^{-1} = 0. \tag{1.4}$$

The corresponding quadratic coefficients of the normal form are, by formulas (V, 3, 2.4),

$$g_1^0 = \varphi_{00}^0 = a_{00}^0, \quad g_1^1 = 2\varphi_{01}^1 = 2a_{01}^1, \quad g_1^{-1} = 2\varphi_{0-1}^{-1} = 2a_{0-1}^{-1}, \tag{1.5}$$

while the remaining quadratic coefficients of the normalizing transformation are given by formulas (V, 3, 2.2)

$$\alpha_{lm}^\nu = \frac{a_{lm}^\nu}{\lambda_l + \lambda_m - \lambda_\nu} \tag{1.6}$$

$$(\nu,\ l,\ m = 0,\ \pm 1;\ \nu,\ l,\ m \neq 0, 0, 0;\ 1, \{0, 1\};\ -1, \{0, -1\}).$$

Repeating this procedure for cubic terms, we set

$$\beta_{000}^0 = \beta_{\{001\}}^1 = \beta_{\{00-1\}}^{-1} = 0 \tag{1.7}$$

and then by means of formulas (V, 3, 2.6) and (V, 3, 2.3) we find

$$g_2^0 = \chi_{000}^0 = b_{000}^0 + 2 \sum_{j=0,\ \pm 1} a_{0j}^0 \alpha_{00}^j,$$

$$g_2^1 = \overline{g_2^{-1}} = 3\chi_{001}^1 = 3b_{001}^1 + 2\sum_j (2a_{0j}^1 \alpha_{01}^j + a_{1j}^1 \alpha_{00}^j), \tag{1.8}$$

$$\beta_{lmp}^\nu = \frac{1}{\lambda_l + \lambda_m + \lambda_p - \lambda_\nu} \left[b_{lmp}^\nu + \frac{2}{3} \sum_j (a_{lj}^\nu \alpha_{mp}^j + a_{mj}^\nu \alpha_{pl}^j + a_{pj}^\nu \alpha_{lm}^j) \right] \tag{1.9}$$

$$(\nu,\ l,\ m,\ p = 0,\ \pm 1;\ \nu,\ l,\ m,\ p \neq 0, 0, 0, 0;$$
$$1,\ \{0, 0, 1\};\ -1,\ \{0, 0. -1\}).$$

Within the second-power terms, the normalizing transformation (1, 1.3) is

$$x_0 = y_0 + 2\,\mathrm{Re}\,(\alpha^0_{11} y_1^2) + 2\alpha^0_{1-1}\,|\,y_1\,|^2 + 4\,\mathrm{Re}\,(\alpha^0_{01} y_0 y_1) + (3),$$

$$x_1 = \overline{x}_{-1} = y_1 + \alpha^1_{00} y_0^2 + \alpha^1_{11} y_1^2 + \alpha^1_{-1-1} \overline{y}_1^2$$
$$+ 2\alpha^1_{0-1} y_0 \overline{y}_1 + 2\alpha^1_{1-1}\,|\,y_1\,|^2 + (3). \qquad (1.10)$$

The coefficients here are given by formulas (1.6) in terms of the coefficients of the initial system (1, 1.1) ($\lambda_0 = 0$, $\lambda_{\pm 1} = -\gamma \pm i\omega$)

$$\alpha^0_{11} = \frac{1}{2}\,\frac{a^0_{11}}{-\gamma + i\omega}, \quad \alpha^0_{1-1} = -\frac{a^0_{1-1}}{2\gamma}, \quad \alpha^0_{01} = \frac{a^0_{01}}{-\gamma + i\omega},$$

$$\alpha^1_{00} = \frac{a^1_{00}}{\gamma - i\omega}, \quad \alpha^1_{11} = \frac{a^1_{11}}{-\gamma + i\omega}, \quad \alpha^1_{-1-1} = -\frac{a^1_{-1-1}}{\gamma + 3i\omega},$$

$$\alpha^1_{0-1} = i\,\frac{a^1_{0-1}}{2\omega}, \quad \alpha^1_{1-1} = -\frac{a^1_{1-1}}{\gamma + i\omega}.$$

Using formulas (1.7) and (1.9), we can extend transformation (1.10) to include third-power terms.

3.2. Integration of normal form. The first equation of (1.3) gives the integral

$$\int_{y_0^0}^{y_0} \frac{dy}{y \sum_1 g^0_s y^s} = t. \qquad (2.1)$$

The second (and third) equations of (1.3) are also solved in quadratures after inverting integral (2.1).

If $g^0_1 = a^0_{00} \neq 0$, the second approximation becomes

$$\int_{y_0^0}^{y_0} \frac{dy}{a^0_{00} y^2} = t, \quad y_0 = \left(\frac{1}{y_0^0} - a^0_{00} t\right)^{-1} \quad (y^0_0 \neq 0),$$

which immediately demonstrates the instability of the trivial solution of system (1.3) and therefore that of the initial system.

If $g^0_1 = 0$, but $g^0_2 \neq 0$ (see (1.8)), the third approximation yields

$$y_0 = \mathrm{sign}\, y^0_0\, (y_0^{0^{-2}} - 2g^0_2 t)^{-1/2}.$$

If $g^0_2 < 0$, then $y_0(t) \to 0$ for $t \to \infty$ and, as follows from the second equation of (1.3), $y_1(t) \to 0$ for $t \to \infty$. The trivial solution of system (1.3), and therefore, of the initial system, is thus asymptotically stable for $g^0_2 < 0$, and unstable for $g^0_2 > 0$.

The above analysis corresponds to the critical case of a single zero root as investigated by Lyapunov ([108a], §§ 28-32). It is also based on the Lyapunov theorem proving stability in the mth approximation in the first critical case ([108a], § 32).

Results on the sensitivity of stability and instability characteristics can be found in Krasovskii's monograph ([100], § 19); there the reader will also find the theorem on stability in the mth approximation (§ 22) and additional results on the critical case of a single zero root (§ 26).

3.3. Remark on convergence. Let us consider the sufficient conditions for convergence of a normalizing transformation (and normal form) stated in the Brjuno theorem (V, 2.4). Equation (1.1) shows that

$$|(\Lambda, Q)| = +\sqrt{\gamma^2 (q_1 + q_{-1})^2 + \omega^2 (q_1 - q_{-1})^2}.$$

By definition (V, 2, 4.1),

$$\omega_k = \Delta = \inf(\gamma, \omega) \quad (k = 1, 2, \ldots).$$

The left-hand side of condition ω (V, 2.4) becomes

$$-\sum_{k=1}^{\infty} \frac{\ln \Delta}{2^k} = -\ln \Delta,$$

and condition ω is obviously satisfied. Condition A' (V, 2.4) becomes

$$\psi_0 \equiv 0, \tag{3.1}$$

where ψ_0 is the right-hand side of the first equation of (1.3). Since in the case under discussion only one eigenvalue ($\lambda_0 = 0$) of the matrix of the linear part of the initial system lies on the imaginary axis, we see that case (1*) of Chapter V, Subsection 2.4 holds, when condition A is reduced to condition A'. Conditions ω and A are therefore satisfied if (3.1) is satisfied or, in detailed form, if

$$g_s^0 = 0 \quad (s = 1, 2, \ldots). \tag{3.2}$$

Consequently, the normalizing transformation discussed in this subsection is convergent in some neighbourhood of zero, provided that all its arbitrary coefficients are set to zero and conditions (3.2) are satisfied.

Under the conditions of this section and of Sections 1 and 2, the sufficient conditions for divergence of Chapter V, Subsection 2.4 fail.

3.4. Free oscillations in a tracking system with a TV sensor. Let us consider a system comprising a gyroscope with two degrees of freedom, to whose inner frame an optical TV device is rigidly fixed [59]. The image of an object tracked in the optical system is projected onto a CRT screen covered by a rectangular raster of light-sensitive point elements. An illuminated spot appears on the raster when the object is within the field of view of the optical system. The object is tracked by means of a special TV sensor [59] with a rec-

tangular window that constantly covers the image of the object (Fig. 13). The TV system superimposes the centre of the window on the centroid of the image on the raster. The tracking system must achieve this with minimum error.

Let us consider the case when the object's image is larger than the tracking window. This situation arises when the distance from the

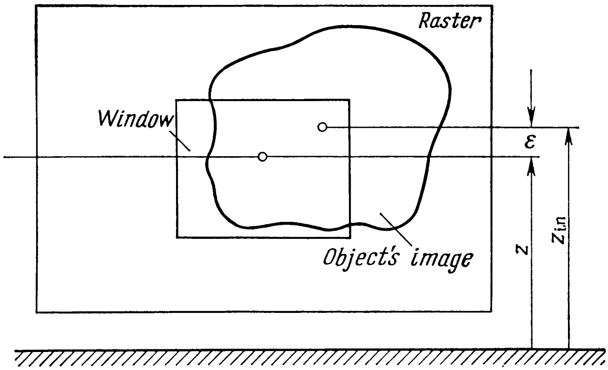

FIG. 13

object is small. Moreover, if the spot covers the window completely, the output of the TV sensor is zero. In this mode the TV tracking system is, therefore, a nonlinear element of the zero-sensitivity-zone type, which will be approximated here by a cubic parabola. The

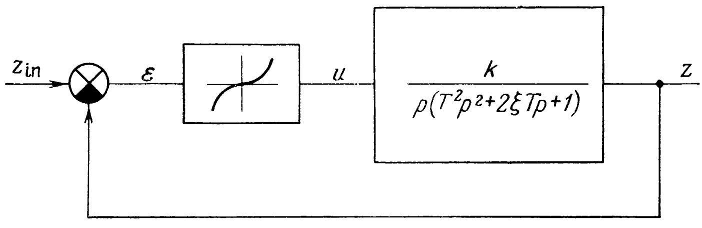

FIG. 14

amplitude-frequency characteristics of the gyrostabilizer, which were recorded on a test unit, will be approximated by an oscillation link with integration. Figure 14 gives the structural diagram of the tracking system in question for the situation described above.

For $z_{\text{in}} = 0$, this diagram corresponds to the equation

$$\dddot{z} + 2a\ddot{z} + b^2\dot{z} = -cz^3,$$

where

$$a = \frac{1}{T}\xi, \quad b = \frac{1}{T}, \quad c = \frac{k}{T^2}$$
$$(0 < \xi < 1,\ k > 0,\ T > 0).$$

Replacement of the independent variable by $\tau = bt$ transforms it to

$$z''' + 2\xi z'' + z' = -\gamma z^3 \quad \left(' = \frac{d}{d\tau}, \quad \gamma = kT\right), \tag{4.1}$$

which can be represented by the system

$$\frac{d\mathbf{z}}{d\tau} = \mathbf{A}\mathbf{z} + \begin{pmatrix} 0 \\ 0 \\ -\gamma z^3 \end{pmatrix}, \quad \mathbf{z} = \begin{pmatrix} z \\ z' \\ z'' \end{pmatrix}, \quad \mathbf{A} = \begin{Vmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & -2\xi \end{Vmatrix}. \tag{4.2}$$

The eigenvalues of the matrix $\mathbf{A}$ are

$$0, \ e^{i\varphi}, \ e^{-i\varphi} \quad (\varphi = \arccos(-\xi)).$$

The matrix $\mathbf{S}$, which reduces $\mathbf{A}$ to diagonal form, has the eigenvectors of $\mathbf{A}$ as its elements. This matrix and its inverse are

$$\mathbf{S} = \begin{Vmatrix} 1 & 1 & 1 \\ 0 & e^{i\varphi} & e^{-i\varphi} \\ 0 & e^{2i\varphi} & e^{-2i\varphi} \end{Vmatrix}, \quad \mathbf{S}^{-1} = \begin{Vmatrix} 1 & 2\xi & 1 \\ 0 & \Delta^{-1}e^{-2i\varphi} & -\Delta^{-1}e^{-i\varphi} \\ 0 & -\Delta^{-1}e^{2i\varphi} & \Delta^{-1}e^{i\varphi} \end{Vmatrix}, \tag{4.3}$$

where $\Delta = \det \mathbf{S} = -2i\sqrt{1-\xi^2}$. The substitution

$$\mathbf{z} = \mathbf{S}\mathbf{x}, \quad \mathbf{x} = \begin{pmatrix} x_0 \\ x_1 \\ x_{-1} \end{pmatrix} \tag{4.4}$$

reduces system (4.2) to diagonal form

$$\frac{d\mathbf{x}}{d\tau} = \operatorname{diag}(0, e^{i\varphi}, e^{-i\varphi})\,\mathbf{x} + \mathbf{S}^{-1}\begin{pmatrix} 0 \\ 0 \\ -\gamma(x_0 + x_1 + x_{-1})^3 \end{pmatrix}. \tag{4.5}$$

Using (4.3), we recast (4.5) into

$$\begin{aligned}
\frac{dx_0}{d\tau} &= -\gamma(x_0 + x_1 + x_{-1})^3, \\
\frac{dx_1}{d\tau} &= \left(-\xi + i\sqrt{1-\xi^2}\right)x_1 + \frac{1}{2}\gamma\left(1 - i\frac{\xi}{\sqrt{1-\xi^2}}\right)(x_0 + x_1 + x_{-1})^3, \\
\frac{dx_{-1}}{d\tau} &= \left(-\xi - i\sqrt{1-\xi^2}\right)x_{-1} + \frac{1}{2}\gamma\left(1 + i\frac{\xi}{\sqrt{1-\xi^2}}\right)(x_0 + x_1 + x_{-1})^3.
\end{aligned} \tag{4.6}$$

Since there are no quadratic terms in (4.6), none appear in normal form (1.2), so that within cubic terms (1.2) is of the form

$$\frac{dy_0}{d\tau} = g_2^0 y_0^3 + (4),$$

$$\frac{dy_1}{d\tau} = \left(-\xi + i\sqrt{1-\xi^2}\right) y_1 + g_2^1 y_0^2 y_1 + (4), \tag{4.7}$$

where g_2^0 and g_2^1 are given by formulas (1.8)

$$g_2^0 = -\gamma, \quad g_2^1 = \frac{3}{2}\gamma\left(1 - i\frac{\xi}{\sqrt{1-\xi^2}}\right).$$

The third equation of (4.7) is omitted because $y_{-1} = \bar{y}_1$ $(x_{-1} = \bar{x}_1)$.

In order to integrate the truncated normal form, we find from the first equation of (4.7)

$$y_0 = y_0^0 (1 + 2\gamma y_0^{0^2}\tau)^{-1/2} \quad (y_0^0 = y_0(0),\ y_1^0 = y_1(0)). \tag{4.8}$$

The second equation of (4.7) then yields

$$\begin{aligned} y_1 = {} & y_1^0 e^{-\xi\tau} (1 + 2\gamma y_0^{0^2}\tau)^{3/4} \left(\cos\sqrt{1-\xi^2}\,\tau + i\sin\sqrt{1-\xi^2}\,\tau\right) \\ & \times \left\{\cos\left[\frac{3}{4}\frac{\xi}{\sqrt{1-\xi^2}}\ln(1 + 2\gamma y_0^{0^2}\tau)\right]\right. \\ & \left. - i\sin\left[\frac{3}{4}\frac{\xi}{\sqrt{1-\xi^2}}\ln(1 + 2\gamma y_0^{0^2}\tau)\right]\right\}. \end{aligned} \tag{4.9}$$

Within cubic terms, the normalizing transformation (1, 1.3) is

$$\begin{aligned} x_\nu = {} & y_\nu + \beta_{000}^\nu y_0^3 + \beta_{111}^\nu y_1^3 + \beta_{-1-1-1}^\nu y_{-1}^3 + 3\beta_{001}^\nu y_0^2 y_1 + 3\beta_{00-1}^\nu y_0^2 y_{-1} \\ & + 3\beta_{011}^\nu y_0 y_1^2 + 3\beta_{0-1-1}^\nu y_0 y_{-1}^2 + 3\beta_{11-1}^\nu y_1^2 y_{-1} + 3\beta_{1-1-1}^\nu y_1 y_{-1}^2 \\ & + 6\beta_{01-1}^\nu y_0 y_1 y_{-1} + (4) \qquad (\nu = 0, \pm 1). \end{aligned}$$

By (4.4) $x_{-1} = \bar{x}_1$, so that the choice $y_{-1} = \bar{y}_1$ was justified; the normalizing transformation can therefore be written (see also (1.7)) as

$$\begin{aligned} x_0 = {} & y_0 + 2\,\mathrm{Re}\,(\beta_{111}^0 y_1^3) + 6y_0^2\,\mathrm{Re}\,(\beta_{001}^0 y_1) + 6y_0\,\mathrm{Re}\,(\beta_{011}^0 y_1^2) \\ & + 6\,|y_1|^2\,\mathrm{Re}\,(\beta_{11-1}^0 y_1) + 6\beta_{01-1}^0 y_0\,|y_1|^2 + (4), \\ x_1 = {} & y_1 + \beta_{000}^1 y_0^3 + \beta_{111}^1 y_1^3 + \beta_{-1-1-1}^1 y_{-1}^3 + 3\beta_{00-1}^1 y_0^2 y_{-1} + 3\beta_{011}^1 y_0 y_1^2 \\ & + 3\beta_{0-1-1}^1 y_0 y_{-1}^2 + 3\beta_{11-1}^1\,|y_1|^2\,y_1 + 3\beta_{1-1-1}^1\,|y_1|^2\,y_{-1} \\ & + 6\beta_{01-1}^1 y_0\,|y_1|^2 + (4). \end{aligned} \tag{4.10}$$

The coefficients here are given by (1.9), and the values of b^{ν}_{lmn} are taken from (4.6); therefore we have

$$\beta^0_{111} = \overline{\beta^0_{-1-1-1}} = -\frac{1}{3}\gamma e^{-i\varphi}, \quad \beta^0_{001} = -\gamma e^{-i\varphi},$$

$$\beta^0_{011} = -\frac{1}{2}\gamma e^{-i\varphi}, \quad \beta^0_{11-1} = -\gamma(2e^{i\varphi} + e^{-i\varphi})^{-1},$$

$$\beta^0_{01-1} = -\gamma(e^{i\varphi} + e^{-i\varphi})^{-1}, \quad \beta^1_{000} = -\frac{\gamma}{\Delta}e^{-2i\varphi},$$

$$\beta^1_{111} = \frac{\gamma}{2\Delta}e^{-2i\varphi}, \quad \beta^1_{-1-1-1} = -\frac{\gamma}{\Delta}(e^{2i\varphi} - 3)^{-1},$$

$$\beta^1_{00-1} = -\frac{\gamma}{\Delta}(e^{2i\varphi} - 1)^{-1}, \quad \beta^1_{011} = \frac{\gamma}{\Delta}e^{-2i\varphi},$$

$$\beta^1_{0-1-1} = -\frac{\gamma}{\Delta}(e^{2i\varphi} - 2)^{-1}, \quad \beta^1_{11-1} = \frac{\gamma}{\Delta}(1 + e^{2i\varphi})^{-1},$$

$$\beta^1_{1-1-1} = \frac{\gamma}{2\Delta}, \quad \beta^1_{01-1} = \frac{\gamma}{\Delta}, \tag{4.11}$$

where, as before,

$$\gamma = kT, \quad e^{i\varphi} = -\xi + i\sqrt{1-\xi^2}, \quad \Delta = -2i\sqrt{1-\xi^2}.$$

Obviously, the inversion of formulas (4.10) reduces to a change in sign of all addends on the right-hand side, beginning with the second. This inversion is required in order to express the initial values y^0_0 and y^0_1 in terms of x^0_0 and x^0_1, and by (4.3) and (4.4), also in terms of z_0, z'_0, and z''_0.

Formulas (4.3) and (4.4) yield

$$z = x_0 + 2\,\mathrm{Re}\,x_1. \tag{4.12}$$

This completes the solution of the Cauchy problem in general form for equation (4.1) within cubic terms. The solution is given as the sequence of formulas (4.12), (4.10), (4.9), and (4.8).

According to the definition given in Subsection 1.4, let us single out the *principal part of the solution.* By (4.10),

$$x_0 = y_0 + (3), \quad x_1 = \overline{x_{-1}} = y_1 + (3),$$

therefore (4.12) yields

$$z = y_0 + 2\,\mathrm{Re}\,y_1 + (3),$$

and the principal part of the solution is, by (4.8) and (4.9),

$$z \approx y^0_0(1 + 2\gamma y^{0\,2}_0\tau)^{-1/2} + 2e^{-\xi\tau}(1 + 2\gamma y^{0\,2}_0\tau)^{3/4} Z(\tau),$$

where

$$Z(\tau) = \cos[\sqrt{1-\xi^2}\,\tau - \Theta(\tau)]\,\mathrm{Re}\,y^0_1 - \sin[\sqrt{1-\xi^2}\,\tau - \Theta(\tau)]\,\mathrm{Im}\,y^0_1$$

and $\Theta(\tau)$ denotes

$$\Theta(\tau)=\frac{3}{4}\frac{\xi}{\sqrt{1-\xi^2}}\ln\left(1+2\gamma y_0^{0^2}\tau\right).$$

By virtue of the definition of the principal part of a solution and by formulas (4.3) and (4.4), we have

$$y_0^0\approx x_0^0=z_0+2\xi z_0'+z_0'',$$

$$y_1^0\approx x_1^0=\Delta^{-1}e^{-2i\varphi}z_0'-\Delta^{-1}e^{-i\varphi}z_0''$$
$$=\left(-\xi+i\frac{2\xi^2-1}{2\sqrt{1-\xi^2}}\right)z_0'+\left(-\frac{1}{2}+i\frac{\xi}{2\sqrt{1-\xi^2}}\right)z_0'',$$

that is,

$$\operatorname{Re}y_1^0\approx-\xi z_0'-\frac{1}{2}z_0'',$$

$$\operatorname{Im}y_1^0\approx\frac{1}{2\sqrt{1-\xi^2}}[(2\xi^2-1)z_0'+\xi z_0''].$$

CHAPTER VIII

NORMAL FORMS OF FOURTH- AND SIXTH-ORDER SYSTEMS IN NEUTRAL LINEAR APPROXIMATION

§ 1. Fourth-Order Systems

The first subsection treats systems of arbitrary order. The second and third subsections, based on the results of Chapter V, analyze resonances and normal forms of real analytic autonomous (in general, nonconservative) fourth-order systems with two pairs of distinct pure imaginary eigenvalues of the matrix of the linear part. Stability of the trivial solution is analyzed in Subsection 1.4 on the basis of either the Molchanov criterion [329b] or, if it fails, the Bibikov-Pliss criterion [227].

1.1. Remark on coefficients of systems of diagonal form. We consider the oscillations in a system with k degrees of freedom that is described by the vector equation

$$\ddot{\mathbf{v}} + \mathbf{P}\mathbf{v} = \mathbf{f}(\mathbf{v}, \dot{\mathbf{v}}), \tag{1.1}$$

where $\mathbf{v} = (v_1, \ldots, v_k)^\tau$, $\mathbf{P}$ is a real symmetric $k \times k$ matrix, and $\mathbf{f}$ is a vector-function analytic in some neighbourhood of the zeros of its arguments. We assume that the eigenvalues of the matrix $\mathbf{P}$, which are real, are positive, and denote them by $\omega_\varkappa^2$ ($\omega_\varkappa > 0$; $\varkappa = 1, \ldots, k$). Let $\mathbf{S}$ be a nonsingular matrix (it can be chosen orthogonal) that reduces $\mathbf{P}$ to diagonal form, that is,

$$\mathbf{P} = \mathbf{S}\,\mathrm{diag}\,(\omega_1^2, \ldots, \omega_k^2)\,\mathbf{S}^{-1}.$$

The linear transformation $\mathbf{v} = \mathbf{S}\mathbf{u}$ reduces system (1.1) to

$$\ddot{u}_\varkappa + \omega_\varkappa^2 u_\varkappa = \sum A^\varkappa_{\alpha\beta} u_\alpha u_\beta + \sum B^\varkappa_{\alpha\beta\gamma} u_\alpha u_\beta u_\gamma + \sum C^\varkappa_{\alpha\beta} \dot{u}_\alpha \dot{u}_\beta$$
$$+ \sum D^\varkappa_{\alpha\beta\gamma} \dot{u}_\alpha \dot{u}_\beta \dot{u}_\gamma + \sum E^\varkappa_{\alpha\beta} u_\alpha \dot{u}_\beta + \sum F^\varkappa_{\alpha\beta\gamma} u_\alpha u_\beta \dot{u}_\gamma$$
$$+ \sum G^\varkappa_{\alpha\beta\gamma} u_\alpha \dot{u}_\beta \dot{u}_\gamma + (4) \quad (\varkappa = 1, \ldots, k), \tag{1.2}$$

where the terms with fourth and higher powers of the variables $u_1, \ldots, u_k, \dot{u}_1, \ldots, \dot{u}_k$ are denoted by (4). Let us rewrite (1.2) as a system of $2k$ first-order equations and reduce the linear part of

the system to diagonal form by the linear substitution (see (VI, 1, 1.2) and (VI, 1, 1.3))

$$x_j = u_{|j|} - i\frac{1}{\omega_j}\dot{u}_{|j|} \quad (i=\sqrt{-1},\ j=\mp 1, \ldots, \mp k), \tag{1.3}$$

$$u_\varkappa = \frac{1}{2}(x_{-\varkappa}+x_\varkappa) = \operatorname{Re} x_\varkappa,$$

$$\dot{u}_\varkappa = \frac{1}{2} i\omega_\varkappa (x_\varkappa - x_{-\varkappa}) = -\omega_\varkappa \operatorname{Im} x_\varkappa \quad (\varkappa = 1, \ldots, k), \tag{1.4}$$

where $\omega_{-\varkappa} = -\omega_\varkappa$ $(\varkappa = 1, \ldots, k)$. Obviously,

$$x_{-\varkappa} = \bar{x}_\varkappa \quad (\varkappa = 1, \ldots, k). \tag{1.5}$$

Manipulations yield

$$\frac{dx_\nu}{dt} = i\omega_\nu x_\nu + \sum_{-k}^{k} a^\nu_{jh} x_j x_h + \sum_{-k}^{k} b^\nu_{jhl} x_j x_h x_l + \ldots \tag{1.6}$$
$$(\nu = \mp 1, \ldots, \mp k).$$

Let us distinguish between the following cases:

(a) If the right-hand side of (1.2) contains only expansions in $u_1, \ldots, u_k$ (i.e. for the second- and third-power terms it consists only of the first and second sums), then the coefficients a^ν_{jh}, $b^\nu_{jhl}, \ldots$ in (1.6) are *pure imaginary* [227].

(b) If the right-hand side of (1.2) contains only expansions in $\dot{u}_1, \ldots, \dot{u}_k$ (i.e. for the second- and third-power terms it includes only the third and fourth sums), then the coefficients of even-power terms in (1.6) (for example, a^ν_{jh}) are pure imaginary, and those of odd-power terms (for example, b^ν_{jhl}) are real.

(c) The presence of only cross terms (the last three sums in (1.2)) results in real coefficients for the quadratic terms in (1.6); the sixth sum in (1.2) also results in real coefficients for third-power terms in (1.6), while the last sum in (1.2) gives pure imaginary coefficients for third-power terms in (1.6).

1.2. Reduction to normal form. We consider a real autonomous fourth-order system under the assumption that the linear part of the system, which has two pairs of distinct pure imaginary conjugate eigenvalues, is reduced to diagonal form

$$\frac{dx_\nu}{d\tau} = \lambda_\nu x_\nu + \sum a^\nu_{jh} x_j x_h + \sum b^\nu_{jhk} x_j x_h x_k + \ldots \tag{2.1}$$
$$(\nu = \mp 1, \mp 2).$$

The values assumed by summation indices are always ± 1, ± 2; $\lambda_{-\nu} = \bar{\lambda}_\nu$; without limiting the general character of the analysis,

we assume that $\lambda_{-1} = -i$, $\lambda_1 = i$, $\lambda_{-2} = -\lambda i$, and $\lambda_2 = \lambda i$ $(0 < \lambda < 1)$; the coefficients a^{ν}_{jh}, b^{ν}_{jhl}, . . . are, in general, complex (see Subsection 1.1) and (we emphasize) symmetrized

$$a^{\nu}_{hj} = a^{\nu}_{jh}, \quad b^{\nu}_{\{jhl\}} = \text{id.}, \tag{2.2}$$

where, as before, $\{\alpha\beta \ldots \omega\}$ stands for any permutation of $\alpha, \beta, \ldots$ $\ldots, \omega$.

By the fundamental Brjuno theorem (see V, 1.2), there exists a reversible complex substitution of variables (normalizing transformation)

$$x_j = y_j + \sum \alpha^j_{lm} y_l y_m + \sum \beta^j_{lmn} y_l y_m y_n + \ldots \tag{2.3}$$
$$(j = \mp 1, \ \mp 2; \quad \alpha^j_{ml} = \alpha^j_{lm}, \ \beta^j_{\{lmn\}} = \text{id.})$$

that reduces system (2.1) to normal form

$$\frac{dy_\nu}{d\tau} = \lambda_\nu y_\nu + y_\nu \sum_{(\Lambda, Q)=0} g_{\nu Q} y_{-1}^{q_{-1}} y_1^{q_1} y_{-2}^{q_{-2}} y_2^{q_2} \tag{2.4}$$
$$(\nu = \mp 1, \ \mp 2).$$

Here

$$Q = (q_{-1}, q_1, q_{-2}, q_2)^\tau,$$
$$(\Lambda, Q) \equiv \lambda_{-1} q_{-1} + \lambda_1 q_1 + \lambda_{-2} q_{-2} + \lambda_2 q_2$$
$$= i\,[(q_1 - q_{-1}) + \lambda\,(q_2 - q_{-2})],$$

q_{-1}, q_1, q_{-2}, and q_2 are integers or zeros, and $q_\nu \geqslant -1$, while the remaining q_j are nonnegative $(q_{-1} + q_1 + q_{-2} + q_2 \geqslant 1)$. Hence, the normal form includes only the *resonant terms* whose exponents satisfy the *resonant equation*

$$(\Lambda, Q) = 0, \quad \text{that is,} \quad q_1 - q_{-1} + \lambda\,(q_2 - q_{-2}) = 0. \tag{2.5}$$

We wish to determine whether (2.4) contains resonant terms of degree r for which

$$q_{-1} + q_1 + q_{-2} + q_2 = r - 1 \quad (r \geqslant 2). \tag{2.6}$$

For any $\lambda \in (0, 1)$ equation (2.5) has the *trivial solution*

$$q_{-1} = q_1, \quad q_{-2} = q_2 \tag{2.7}$$

and the *nontrivial solution*

$$\lambda = \frac{q_{-1} - q_1}{q_2 - q_{-2}}. \tag{2.8}$$

Owing to conditions $0 < \lambda < 1$ and (2.6), relationship (2.8) entails the restrictions

$$0 < |q_{-1} - q_1| < |q_2 - q_{-2}| \leqslant r. \tag{2.9}$$

The trivial solution is impossible for *quadratic terms* ($r = 2$) (as well as for all even-power terms), since by virtue of (2.6) and (2.7) the sum of two even numbers cannot equal an odd number. The nontrivial solution is only possible for $\lambda = \frac{1}{2}$, which gives for the resonant terms of equations (2.4)

$$Q_{-1} = (-1,\ 0,\ 2,\ 0),\quad Q_1 = (0,\ -1,\ 0,\ 2),$$
$$Q_{-2} = (1,\ 0,\ -1,\ 1),\quad Q_2 = (0,\ 1,\ 1,\ -1).$$

For *cubic terms* ($r = 3$) the trivial solution (2.7) yields

$$Q_\nu = (1,\ 1,\ 0,\ 0) \text{ and } Q_\nu = (0,\ 0,\ 1,\ 1) \quad (\nu = \mp 1,\ \mp 2).$$

The nontrivial solution is possible only for $\lambda = \frac{1}{3}$, which gives for the resonant terms, in addition to the trivial solution,

$$Q_{-1} = (-1,\ 0,\ 3,\ 0),\quad Q_1 = (0,\ -1,\ 0,\ 3),$$
$$Q_{-2} = (1,\ 0,\ -1,\ 2),\quad Q_2 = (0,\ 1,\ 2,\ -1).$$

Conclusion. If equations (2.4) are truncated within the third-power terms, then the fundamental Brjuno theorem (V, 1.2) yields the following normal forms:

(a) With no principal resonances, that is, for $\lambda \neq \frac{1}{2}, \frac{1}{3}$, (i.e. the *general case*)

$$\frac{dy_\nu}{d\tau} = \lambda_\nu y_\nu + g_\nu y_\nu^2 y_{-\nu} + h_\nu y_\nu y_{3-|\nu|} y_{|\nu|-3} \quad (\nu = \mp 1,\ \mp 2). \tag{2.10}$$

(b) *For the first principal resonance* $\left(\lambda = \frac{1}{2}\right)$ each equation of (2.10) must be supplemented on the right by one quadratic term, namely,

$$f_{-1} y_{-2}^2,\quad f_1 y_2^2,\quad f_{-2} y_{-1} y_2,\quad f_2 y_1 y_{-2}, \tag{2.11}$$

respectively.

(c) *For the second principal resonance* $\left(\lambda = \frac{1}{3}\right)$ each equation of (2.10) must be supplemented on the right by one cubic term, namely,

$$e_{-1} y_{-2}^3,\quad e_1 y_2^3,\quad e_{-2} y_{-1} y_2^2,\quad e_2 y_1 y_{-2}^2, \tag{2.12}$$

respectively.

1.3. Calculation of coefficients of normalizing transformation and normal forms. The normalizing transformation (2.3) reduces system (2.1) to normal form (see (2.10)-(2.12)), which, after symmetrization (see (V, 3, 1.3a)), yields the fundamental identities (V, 3, 1.6). Next we follow the alternative of Chapter V, Subsection 3.2.

Suppose first that $\lambda \neq \frac{1}{2}$. Then $\lambda_\nu \neq \lambda_l + \lambda_m$ (ν, l, and $m = \pm 1, \pm 2$), and formula (V, 3, 2.2) holds for quadratic coefficients of the normalizing transformation

$$\alpha_{lm}^{\nu} = \frac{a_{lm}^{\nu}}{\lambda_l + \lambda_m - \lambda_\nu} \quad \left(\nu,\ l,\ m = \mp 1,\ \mp 2;\ \lambda \neq \frac{1}{2}\right) \tag{3.1}$$

with no additional restrictions $\left(\lambda \in \left(0, \frac{1}{2}\right), \left(\frac{1}{2}, 1\right)\right)$.

For $\lambda = \frac{1}{2}$ we obtain that $\lambda_\nu = \lambda_l + \lambda_m$ if and only if ν, l, $m = -1, -2, -2;\ 1, 2, 2;\ -2, \{-1, 2\};\ 2, \{1, -2\}$. We choose

$$\alpha_{-2-2}^{-1}, \quad \alpha_{22}^{1}, \quad \alpha_{\{-12\}}^{-2}, \quad \alpha_{\{1-2\}}^{2}$$

arbitrarily (it is preferable to determine them from continuity by means of (3.1) for $\lambda \to \frac{1}{2}$, if this is possible, or set them to zero). Formula (3.1) holds for the remaining α_{lm}^{ν}. The coefficients of the quadratic terms appearing in (2.11) for $\lambda = \frac{1}{2}$ are determined from formulas (V, 3, 2.4)

$$\begin{aligned} f_{-1} &= \varphi_{-2-2}^{-1} = a_{-2-2}^{-1}, & f_1 &= \varphi_{22}^{1} = a_{22}^{1}, \\ f_{-2} &= 2\varphi_{-12}^{-2} = 2a_{-12}^{-2}, & f_2 &= 2\varphi_{1-2}^{2} = 2a_{1-2}^{2}. \end{aligned} \tag{3.2}$$

Now we consider cubic terms. We assume first that $\lambda \neq \frac{1}{3}$. We select those values of ν, l, m, and p for which $\lambda_\nu = \lambda_l + \lambda_m + \lambda_p$; these are

$$\nu,\ l,\ m,\ p = \nu,\ \{l,\ -l,\ \nu\} \quad (\nu,\ l = \mp 1,\ \mp 2). \tag{3.3}$$

We set

$$\beta_{\{l,\ -l,\ \nu\}}^{\nu} = 0 \quad (\nu,\ l = \mp 1,\ \mp 2). \tag{3.4}$$

Formula (V, 3, 2.3) holds for the remaining values of ν, l, m, and p

$$\beta_{lmp}^{\nu} = \frac{1}{\lambda_l + \lambda_m + \lambda_p - \lambda_\nu}\left[b_{lmp}^{\nu} + \frac{2}{3}\sum_{j=\mp 1,\ \mp 2}\left(a_{jl}^{\nu}\alpha_{mp}^{j} + a_{jm}^{\nu}\alpha_{pl}^{j} + a_{jp}^{\nu}\alpha_{lm}^{j}\right)\right] \tag{3.5}$$

$$(\nu,\ l,\ m,\ p = \mp 1,\ \mp 2;\quad l,\ m,\ p \neq \{l,\ -l,\ \nu\}).$$

Since in general $l = \mp \nu,\ \mp(3 - |\nu|)$, the coefficients of the third-power terms in (2.10) that correspond to the indicated values of the

subscripts and to $\lambda \neq \frac{1}{2}$ are given by formulas (V, 3, 2.6)

$$g_\nu = 3\chi^\nu_{\nu\nu-\nu} = 3b^\nu_{\nu\nu-\nu} + 2 \sum_{j=\mp 1,\ \mp 2} (2a^\nu_{\nu j}\alpha^j_{\nu-\nu} + a^\nu_{-\nu j}\alpha^j_{\nu\nu}),$$

$$h_\nu = 6\chi^\nu_{\nu,\ 3-|\nu|,\ |\nu|-3} = 6b^\nu_{\nu,\ 3-|\nu|,\ |\nu|-3}$$
$$+ 4 \sum_{j=\mp 1,\ \mp 2} (a^\nu_{\nu j}\alpha^j_{3-|\nu|,\ |\nu|-3} + a^\nu_{3-|\nu|,\ j}\alpha^j_{|\nu|-3,\ \nu} + a^\nu_{|\nu|-3,\ j}\alpha^j_{\nu,\ 3-|\nu|}) \quad (3.6)$$
$$\left(\nu = \mp 1,\ \mp 2;\ \lambda \neq \frac{1}{2}\right).$$

Formulas (V, 3, 2.5) must be used for $\lambda = \frac{1}{2}$.

Finally, we consider case (c) of Subsection 1.2, that is, $\lambda = \frac{1}{3}$. In addition to (3.3), $\lambda_\nu = \lambda_l + \lambda_m + \lambda_p$ when ν, l, m, p = —1, —2, —2, —2; 1, 2, 2, 2; —2, {—1, 2, 2}; 2, {1, —2, —2}. We choose

$$\beta^{-1}_{-2-2-2},\quad \beta^1_{222},\quad \beta^{-2}_{\{-122\}},\quad \beta^2_{\{1-2-2\}}$$

arbitrarily (it is preferable to determine them from continuity by means of (3.5) for $\lambda \to \frac{1}{3}$, if this is possible, or set them to zero). The coefficients of the cubic terms appearing in (2.12) for $\lambda = \frac{1}{3}$ are determined from formulas (V, 3, 2.6)

$$e_{-1} = \chi^{-1}_{-2-2-2} = b^{-1}_{-2-2-2} + 2 \sum_{j=\mp 1,\ \mp 2} a^{-1}_{-2j}\alpha^j_{-2-2},$$
$$e_1 = \chi^1_{222} = b^1_{222} + 2 \sum_j a^1_{2j}\alpha^j_{22},$$
$$e_{-2} = 3\chi^{-2}_{-122} = 3b^{-2}_{-122} + 2 \sum_j (a^{-2}_{-1j}\alpha^j_{22} + 2a^{-2}_{2j}\alpha^j_{-12}),$$
$$e_2 = 3\chi^2_{1-2-2} = 3b^2_{1-2-2} + 2 \sum_j (a^2_{1j}\alpha^j_{-2-2} + 2a^2_{-2j}\alpha^j_{1-2}). \quad (3.7)$$

For any $\lambda \in (0, 1)$, the calculations start with formulas (3.6) (but with (3.7) if $\lambda = \frac{1}{3}$ and then (3.6)) and are completed with formulas (3.5).

1.4. The Molchanov criterion of oscillation stability. Kamenkov [83] and Malkin [111c] investigated stability in the critical case of two pairs of pure imaginary roots. In this chapter it is more convenient to use the Molchanov criterion [329b], as it reflects the specifics of normal forms and the accompanying resonances.

Now* we assume, as in Subsection 4.1, that the variables are pairwise conjugate

$$x_{-j} = \bar{x}_j, \quad y_{-j} = \bar{y}_j \quad (j = 1, 2).$$

We consider the general case $\left(\lambda \neq \frac{1}{2}, \frac{1}{3} \right)$. Multiplying equations (2.10) by the corresponding $\bar{y}_\nu$ and adding them pairwise, we obtain

$$\frac{d\,|y_1|^2}{d\tau} = 2\operatorname{Re} g_1 |y_1|^4 + 2\operatorname{Re} h_1\, |y_1|^2\, |y_2|^2,$$

$$\frac{d\,|y_2|^2}{d\tau} = 2\operatorname{Re} h_2 |y_1|^2\, |y_2|^2 + 2\operatorname{Re} g_2\, |y_2|^4. \tag{4.1}$$

By definition, instability is established if we find one unstable trajectory. Assuming $y_2 \equiv 0$ and then $y_1 \equiv 0$, we obtain that when either of the two inequalities

$$\operatorname{Re} g_1 > 0 \quad \text{or} \quad \operatorname{Re} g_2 > 0 \tag{4.2}$$

is satisfied, the trivial solution of system (4.1) is unstable.

We assume henceforth that

$$\operatorname{Re} g_1 < 0, \quad \operatorname{Re} g_2 < 0.$$

The case $\operatorname{Re} g_1 = \operatorname{Re} g_2 = 0$ will be considered in the next subsection. With the new variables of fixed sign (nonnegative)

$$v_1 = -2\operatorname{Re} g_1\, |\, y_1\, |^2, \quad v_2 = -2\operatorname{Re} g_2\, |\, y_2\, |^2$$

system (4.1) is rewritten in the form

$$\frac{dv_1}{d\tau} = -v_1(v_1 + av_2), \quad \frac{dv_2}{d\tau} = -v_2(bv_1 + v_2), \tag{4.3}$$

where

$$a = \frac{\operatorname{Re} h_1}{\operatorname{Re} g_2}, \quad b = \frac{\operatorname{Re} h_2}{\operatorname{Re} g_1}$$

(we recall that g_1, g_2, h_1, and h_2 are given by (3.6)).

The Molchanov criterion [329b]. *Unstable systems* (4.3) *lie below the negative branch of the hyperbola* $ab = 1$. *The region of monotonic stability lies above the straight line* $a + b = -2$.

Proof. Summing equations (4.3), we obtain

$$\frac{d(v_1 + v_2)}{d\tau} = (\mathbf{A}\mathbf{v}, \mathbf{v}),$$

where

$$\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \quad \mathbf{A} = \begin{Vmatrix} -1 & -\frac{1}{2}(a+b) \\ -\frac{1}{2}(a+b) & -1 \end{Vmatrix}.$$

* This assumption is unnecessary in Subsections 1.2 and 1.3.

We recall that v_1 and v_2 are nonnegative variables. Therefore the conditions for the negative definiteness of the matrix **A** are sufficient conditions for asymptotic stability of the trivial solution of system (4.3). These last conditions result in the inequalities

$$-2 < a + b \quad \text{and} \quad a + b < 2. \tag{4.4}$$

Let us demonstrate that the second condition of (4.4) can be dropped. We consider the function

$$V = \frac{1}{2}(v_1 + v_2)^2,$$

which is positive-definite for the nonnegative variables v_1 and v_2. By (4.3), its derivative is equal to

$$\frac{dV}{d\tau} = -v_1^3 - (a + b + 1)\, v_1 v_2 (v_1 + v_2) - v_2^3$$

and is negative-definite if

$$a + b + 1 > 0. \tag{4.5}$$

By the Lyapunov theorem on asymptotic stability under condition (4.5), the trivial solution of system (4.3) is asymptotically stable. Conditions (4.4) and (4.5) reduce to a single inequality

$$a + b > -2,$$

which proves the second half of the criterion.

Let us prove the first half. We seek solutions of system (4.3) in the form

$$v_\varkappa = v_\varkappa^0 v(\tau) \quad (v_\varkappa^0 > 0, \quad \varkappa = 1,\ 2). \tag{4.6}$$

Substitution of these expressions into (4.3) yields

$$\frac{dv}{d\tau} = \sigma v^2, \quad v(0) = 1,$$

$$-v_1^0 - a v_2^0 = \sigma, \quad -b v_1^0 - v_2^0 = \sigma,$$

whence

$$v_1^0 = \frac{\sigma(a-1)}{1-ab}, \quad v_2^0 = \frac{\sigma(b-1)}{1-ab}.$$

If $\sigma > 0$, solution (4.6) tends to infinity when $\tau \to \infty$. Since $v_1^0 > 0$ and $v_2^0 > 0$, the condition $\sigma > 0$ holds if

$$ab > 1, \quad a < 1, \quad b < 1,$$

which completes the proof of the first statement of the criterion.

Comment. Molchanov also asserts that the region between the negative branch of the hyperbola $ab = 1$ and the straight line $a + b = -2$ contains formally stable systems (4.3) whose solutions

may, however, increase before damping out [329b]. The degree of "amplification" in such systems can be roughly evaluated by $|a| + |b|$. Clearly, these systems may prove practically unstable in cases of great "amplification".

1.5. The Bibikov-Pliss criterion. The coefficients g_ν and h_ν ($\nu = \mp 1, \mp 2$) of system (2.10) in case (a) of Subsection 1.1 are pure imaginary, that is, the Molchanov criterion definitely fails. Let us rewrite the second and fourth equations of system (2.10) $\left(\lambda \in (0, 1); \lambda \neq \frac{1}{2}, \frac{1}{3}\right)$ in the form

$$\frac{dy_1}{d\tau} = iy_1 + i\left(\frac{g_1}{i} y_1 |y_1|^2 + \frac{h_1}{i} y_1 |y_2|^2\right) + (4),$$
$$\frac{dy_2}{d\tau} = i\lambda y_2 + i\left(\frac{h_2}{i} y_2 |y_1|^2 + \frac{g_2}{i} y_2 |y_2|^2\right) + (4). \tag{5.1}$$

System (5.1) is a special case of system (1.2).

Applied to system (5.1), the Bibikov-Pliss theorem [227] leads to the following **criterion**: *If* λ *is an irrational number and* (see (3.6))

$$J = \begin{vmatrix} \frac{1}{i} g_1 & \frac{1}{i} h_1 \\ \frac{1}{i} h_2 & \frac{1}{i} g_2 \end{vmatrix} = -(g_1 g_2 - h_1 h_2) \neq 0, \tag{5.2}$$

then "almost all" of an ε*-neighbourhood of the origin of the reference frame is filled with almost periodic motions, that is, the position of equilibrium is "practically" stable in the sense of Lyapunov.*

§ 2. The Ishlinskii Problem

The presentation in Subsections 2.4-2.6 is independent of that in 2.2 and 2.3.

2.1. Reduction of equations of motion to the Lyapunov form. Ishlinskii ([79a], Appendix 2) derived exact (in terms of precession theory) equations of motion (40)* for the sensor frame of a gyrohorizon compass with respect to Darboux's trihedron. Under assumptions (17), (41), and (47), and an additional assumption concerning the uniform motion of the suspension point on a sphere, these equations are

$$\frac{d\alpha}{dt} = \frac{mlv\omega}{2B\cos\varepsilon}\cos\alpha\cos\gamma + \frac{mgl}{2B\cos\varepsilon}\tan\beta\cos\gamma + \frac{v}{R}\cos\alpha\tan\beta - \omega,$$

$$\frac{d\beta}{dt} = \frac{mlv\omega}{2B\cos\varepsilon}\cos\alpha\cos\beta\sin\gamma + \frac{mgl}{2B\cos\varepsilon}\sin\beta\sin\gamma - \frac{v}{R}\sin\alpha,$$

* Formulas cited by single numbers refer to [79a].

$$\frac{d\gamma}{dt}=\frac{2B\cos\varepsilon}{mlR}-\frac{mlv\omega}{2B\cos\varepsilon}\cos\alpha\sin\beta\cos\gamma$$
$$-\frac{mgl}{2B\cos\varepsilon}\frac{\sin^2\beta}{\cos\beta}\cos\gamma-\frac{v}{R}\frac{\cos\alpha}{\cos\beta},$$
$$\frac{d\varepsilon}{dt}=-\frac{mlv\omega}{2B\sin\varepsilon}(\sin\alpha\cos\gamma+\cos\alpha\sin\beta\sin\gamma)+\frac{mgl}{2B\sin\varepsilon}\cos\beta\sin\gamma. \tag{1.1}$$

Note that system (1.1) is reversible (t-invariant), that is, it is not altered by the replacement of α by $-\alpha$, γ by $-\gamma$, and t by $-t$. As in expressions (45), (19), and (20), we set in (1.1)

$$\varepsilon=\varepsilon_0+\frac{mlv}{2B\sin\varepsilon_0}\Delta\quad\left(\cos\varepsilon_0=\frac{mlv}{2B}<1\right) \tag{1.2}$$

and, truncating equations (1.1) within the third-power terms, we recast system (1.1) to

$$\frac{d\alpha}{dt}=N\mathrm{B}+\omega\Delta-\frac{1}{2}\omega\alpha^2+\omega\left(1+\frac{1}{2}H\right)\Delta^2+N\mathrm{B}\Delta$$
$$-\frac{1}{2}\omega\alpha^2\Delta+\omega\left(1+\frac{5}{6}H\right)\Delta^3+N\left(1+\frac{1}{2}H\right)\mathrm{B}\Delta^2,$$
$$\frac{d\mathrm{B}}{dt}=-N\alpha+\omega\Gamma+N\mathrm{B}\Gamma+\omega\Gamma\Delta$$
$$+\frac{1}{6}N\alpha^3-\frac{1}{2}\omega\alpha^2\Gamma+\omega\left(1+\frac{1}{2}H\right)\Gamma\Delta^2+N\mathrm{B}\Gamma\Delta,$$
$$\frac{d\Gamma}{dt}=-\omega\mathrm{B}-N\Delta+\frac{1}{2}N\alpha^2-N\mathrm{B}^2-\frac{1}{2}NH\Delta^2-\omega\mathrm{B}\Delta$$
$$+\frac{1}{6}NH\Delta^3+\frac{1}{2}\omega\alpha^2\mathrm{B}-\omega\left(1+\frac{1}{2}H\right)\mathrm{B}\Delta^2-N\mathrm{B}^2\Delta,$$
$$\frac{d\Delta}{dt}=-\omega\alpha+N\Gamma+\omega H\alpha\Delta-\omega\frac{V^2}{N^2}\mathrm{B}\Gamma-NH\Gamma\Delta$$
$$-\omega\left(\frac{1}{2}H+H^2\right)\alpha\Delta^2+\frac{1}{6}\omega\alpha^3+NH\left(\frac{1}{2}+H\right)\Gamma\Delta^2. \tag{1.3}$$

The parameters introduced here are

$$N^2=\frac{g}{R},\quad V=\frac{v}{R},\quad H=\frac{m^2l^2v^2}{4B^2-m^2l^2v^2}=\cot^2\varepsilon_0 \tag{1.4}$$

and the variables are

$$\alpha,\quad \mathrm{B}=\frac{N}{V}\beta,\quad \Gamma=\frac{N}{V}\gamma,\quad \Delta. \tag{1.5}$$

Furthermore, here and henceforth we neglect V^2 as compared to N^2

$$\frac{V^2}{N^2}\equiv\frac{v^2}{gR}\ll 1. \tag{1.6}$$

This means that for velocities of the suspension point below 100 m/s we neglect values smaller than 1.5×10^{-4} as compared to unity. The eigenvalues of the matrix **A** of the linear part of system (1.3) are

$$\mp(N+\omega)\,i, \quad \mp|N-\omega|\,i \quad (i=\sqrt{-1}). \tag{1.7}$$

The matrix $\mathbf{S}_1$ of the linear transformation that reduces the linear part of system (1.3) to diagonal form has the eigenvectors of **A** as its elements and is equal to

$$\mathbf{S}_1 = \begin{Vmatrix} 1 & 1 & 1 & 1 \\ -i & i & -i & i \\ -1 & -1 & 1 & 1 \\ -i & i & i & -i \end{Vmatrix}. \tag{1.8}$$

We can readily obtain the matrix $\mathbf{S}_2$ of the linear transformation that reduces the diagonal system to the Lyapunov skew-symmetric form

$$\mathbf{S}_2 = \frac{1}{2}\begin{Vmatrix} 1 & -i & 0 & 0 \\ 1 & i & 0 & 0 \\ 0 & 0 & 1 & -i \\ 0 & 0 & 1 & i \end{Vmatrix}.$$

The matrix **S** of the resultant transformation is then

$$\mathbf{S} = \mathbf{S}_1\mathbf{S}_2 = \begin{Vmatrix} 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{Vmatrix}.$$

Now we can make a linear change of variables in system (1.3) (the inverse transformation is given in parentheses)

$$\alpha = \xi + \zeta, \quad \mathrm{B} = -\eta - \chi, \quad \Gamma = \zeta - \xi, \quad \Delta = \chi - \eta \tag{1.9}$$

$$\left(\xi = \frac{1}{2}(\alpha - \Gamma), \quad \eta = -\frac{1}{2}(\mathrm{B}+\Delta), \right.$$

$$\left.\zeta = \frac{1}{2}(\alpha+\Gamma), \quad \chi = \frac{1}{2}(\Delta - \mathrm{B})\right).$$

Under assumption (1.6), equation of motion (1.1), in which we retain only second-power terms, can be rewritten in new (again dimensionless) variables as the system

$$\frac{d\xi}{d\tau} = -\eta - \frac{1}{4}(\xi+\zeta)^2 + \left(1+\frac{1}{4}H\right)\eta^2 + \frac{1}{4}H\chi^2 + \left(\lambda - \frac{1}{2}H\right)\eta\chi,$$

$$\frac{d\eta}{d\tau} = \xi + \frac{1}{2}[(H-1)\xi\eta - (H+\lambda)\xi\chi + (1-\lambda H)\eta\zeta + \lambda(1+H)\zeta\chi],$$

$$\frac{d\zeta}{d\tau} = -\lambda\chi + \frac{1}{4}\lambda(\xi+\zeta)^2 - \frac{1}{4}\lambda H\eta^2 - \lambda\left(1+\frac{1}{4}H\right)\chi^2 + \left(\frac{1}{2}\lambda H - 1\right)\eta\chi,$$

$$\frac{d\chi}{d\tau} = \lambda\zeta + \frac{1}{2}[-(1+H)\xi\eta + (H-\lambda)\xi\chi + (1+\lambda H)\eta\zeta + \lambda(1-H)\zeta\chi]. \quad (1.10)$$

Here we have introduced dimensionless time

$$\tau = (N+\omega)\, t \quad (1.11)$$

and two dimensionless parameters: H (see (1.4)) and

$$\lambda = \frac{N-\omega}{N+\omega} \quad (0<\lambda<1) \quad (1.12)$$

the case $\omega > N$ can be analyzed if we introduce $\lambda' = -\lambda$). Note that system (1.10) is reversible (t-invariant), that is, it remains unaltered if we replace ξ by $-\xi$, ζ by $-\zeta$, and τ by $-\tau$.

2.2. Transformation of systems similar to Lyapunov. System (1.10) is a Lyapunov system ([108a], §§ 33-45) provided it possesses a first integral, which is analytic and of fixed sign in some neighbourhood of the zeros of its variables. It is easy to check that for $\lambda \in (0, 1)$ the first integral of system (1.10) is of fixed sign, in the approximation stated,

$$G = \xi^2 + \eta^2 + \zeta^2 + \chi^2 + \frac{1}{2}H(\chi^3 - \eta^3) + \frac{1}{2}\eta(\xi^2+\zeta^2-3H\chi^2) - \frac{1}{2}\chi(\xi^2+\zeta^2-3H\eta^2) + \xi\zeta(\eta-\chi) + (4) = \mu^2 \quad (\mu>0). \quad (2.1)$$

Since, however, the convergence of (2.1) was not established, system (1.10) is said to be *similar to Lyapunov.*

Using integral (2.1) and Lyapunov's substitution

$$\xi = \rho\cos\vartheta, \quad \eta = \rho\sin\vartheta, \quad \zeta = \rho z_1, \quad \chi = \rho z_2, \quad (2.2)$$

a Lyapunov system may be reduced to a nonautonomous quasilinear system with order lowered by two (see the transformation given

in Chapter I, Subsection 1.1). We denote quadratic terms (and those of higher power in equations (1.10)) by

$$\Xi(\xi, \eta, \zeta, \chi), \quad \mathrm{H}(\xi, \eta, \zeta, \chi), \quad \mathrm{Z}(\xi, \eta, \zeta, \chi), \quad \mathrm{X}(\xi, \eta, \zeta, \chi)$$

and compute the functions

$$\begin{aligned}
\mathrm{P}(\rho, \vartheta, z_1, z_2) &= \rho^{-2}(\Xi\cos\vartheta + \mathrm{H}\sin\vartheta),\\
\Theta(\rho, \vartheta, z_1, z_2) &= \rho^{-2}(-\Xi\sin\vartheta + \mathrm{H}\cos\vartheta),\\
Z_1(\rho, \vartheta, z_1, z_2) &= \rho^{-2}\mathrm{Z} - z_1\mathrm{P},\\
Z_2(\rho, \vartheta, z_1, z_2) &= \rho^{-2}\mathrm{X} - z_2\mathrm{P}.
\end{aligned}$$

We obtain

$$\begin{aligned}
\mathrm{P} = {} & \frac{1}{8}(2+3H)\sin\vartheta\sin 2\vartheta + \frac{1}{2}(1-\lambda H)z_1\sin^2\vartheta\\
& + \frac{1}{4}(\lambda - 2H)z_2\sin 2\vartheta - \frac{1}{4}(\cos\vartheta + z_1)^2\cos\vartheta\\
& + \frac{1}{2}\lambda(1+H)z_1z_2\sin\vartheta + \frac{1}{4}Hz_2^2\cos\vartheta + O(\rho),\\
\Theta = {} & -\left(1+\frac{1}{4}H\right)\sin^3\vartheta + \frac{1}{4}(H-1)\cos\vartheta\sin 2\vartheta\\
& + \frac{1}{4}(1-\lambda H)z_1\sin 2\vartheta + \left(\frac{1}{2}H - \lambda\right)z_2\sin^2\vartheta\\
& + \frac{1}{2}\lambda(1+H)z_1z_2\cos\vartheta - \frac{1}{4}Hz_2^2\sin\vartheta\\
& - \frac{1}{2}(H+\lambda)z_2\cos^2\vartheta + \frac{1}{4}(\cos\vartheta + z_1)^2\sin\vartheta + O(\rho),\\
Z_1 = {} & -\frac{1}{4}H\lambda\sin^2\vartheta + \left(\frac{1}{2}\lambda H - 1\right)z_2\sin\vartheta\\
& + \frac{1}{4}\lambda(\cos\vartheta + z_1)^2 - \frac{1}{2}(1-\lambda H)z_1^2\sin^2\vartheta\\
& - \frac{1}{8}(2+3H)z_1\sin\vartheta\sin 2\vartheta + \frac{1}{4}(2H-\lambda)z_1z_2\sin 2\vartheta\\
& - \lambda\left(1+\frac{1}{4}H\right)z_2^2 + \frac{1}{4}(\cos\vartheta + z_1)^2 z_1\cos\vartheta\\
& - \frac{1}{2}\lambda(1+H)z_1^2z_2\sin\vartheta - \frac{1}{4}Hz_1z_2^2\cos\vartheta + O(\rho),\\
Z_2 = {} & -\frac{1}{4}(1+H)\sin 2\vartheta + \frac{1}{2}(1+\lambda H)z_1\sin\vartheta\\
& + \frac{1}{2}(H-\lambda)z_2\cos\vartheta - \frac{1}{8}(2+3H)z_2\sin\vartheta\sin 2\vartheta\\
& + \frac{1}{2}\lambda(1-H)z_1z_2 - \frac{1}{2}(1-\lambda H)z_1z_2\sin^2\vartheta
\end{aligned}$$

$$+\frac{1}{4}(2H-\lambda)z_2^2\sin 2\vartheta+\frac{1}{4}(\cos\vartheta+z_1)^2 z_2\cos\vartheta$$
$$-\frac{1}{2}\lambda(1+H)z_1z_2^2\sin\vartheta-\frac{1}{4}Hz_2^3\cos\vartheta+O(\rho). \tag{2.3}$$

The transformation ultimately yields a nonautonomous quasilinear system of the type (I, 1, 1.9)

$$\frac{dz_1}{d\vartheta}=-\lambda z_2+\mu(1+z_1^2+z_2^2)^{-1/2}[Z_1(0,\ \vartheta,\ z_1,\ z_2)$$
$$+\lambda z_2\Theta(0,\ \vartheta,\ z_1,\ z_2)]+O(\mu^2),$$
$$\frac{dz_2}{d\vartheta}=\lambda z_1+\mu(1+z_1^2+z_2^2)^{-1/2}[Z_2(0,\ \vartheta,\ z_1,\ z_2)$$
$$-\lambda z_1\Theta(0,\ \vartheta,\ z_1,\ z_2)]+O(\mu^2). \tag{2.4}$$

Various methods of small parameter may be applied to system (2.4). The one we select in this chapter is the Poincaré method of determining periodic solutions ([188a], vol. I, Ch. III).

2.3. Determination of periodic solutions. We shall seek periodic solutions of system (2.4) in the form

$$z_1(\vartheta)=z_1^0(\vartheta)+\mu z_1^1(\vartheta)+\mu^2z_1^2(\vartheta)+\cdots,$$
$$z_2(\vartheta)=z_2^0(\vartheta)+\mu z_2^1(\vartheta)+\mu^2z_2^2(\vartheta)+\cdots. \tag{3.1}$$

Substitution of these series into (2.4) yields systems of equations for z_1^0, z_2^0, and z_1^1, z_2^1

$$\frac{dz_1^0}{d\vartheta}=-\lambda z_2^0,$$
$$\frac{dz_2^0}{d\vartheta}=\lambda z_1^0; \tag{3.2}$$

$$\frac{dz_1^1}{d\vartheta}=-\lambda z_2^1+(1+z_1^{0^2}+z_2^{0^2})^{-1/2}[Z_1(0,\ \vartheta,\ z_1^0,\ z_2^0)+\lambda z_2^0\Theta(0,\ \vartheta,\ z_1^0,\ z_2^0)],$$
$$\frac{dz_2^1}{d\vartheta}=\lambda z_1^1+(1+z_1^{0^2}+z_2^{0^2})^{-1/2}[Z_2(0,\ \vartheta,\ z_1^0,\ z_2^0)-\lambda z_1^0\Theta(0,\ \vartheta,\ z_1^0,\ z_2^0)]. \tag{3.3}$$

The general solution of system (3.2) is $T(\lambda)$-periodic

$$z_1^0=C\cos\lambda\vartheta+D\sin\lambda\vartheta,$$
$$z_2^0=-D\cos\lambda\vartheta+C\sin\lambda\vartheta. \tag{3.4}$$

Solution (3.4) may also be regarded as $qT(\lambda)$-periodic, where q is any natural number. The right-hand sides of equations (2.4) and (3.3) are explicit functions of the independent variable, and this dependence may be regarded as $2p\pi$-periodic for any natural p. There-

fore solution (3.4) is generating for a $2p\pi$-periodic solution of system (2.4) if and only if

$$qT(\lambda) = 2p\pi \quad \text{or} \quad \lambda = \frac{q}{p}, \tag{3.5}$$

where q/p is any positive irreducible proper fraction. Let us reduce the system of differential equations (3.3) for the first corrections to a single equation in z_1^1; taking into account (3.4), we obtain

$$\frac{d^2 z_1^1}{d\vartheta^2} + \lambda^2 z_1^1 = (1 + C^2 + D^2)^{-1/2} \Big[\frac{d}{d\vartheta} Z_1(0, \vartheta, z_1^0(\vartheta), z_2^0(\vartheta))$$
$$+ \lambda z_2^0 \frac{d}{d\vartheta} \Theta(0, \vartheta, z_1^0(\vartheta), z_2^0(\vartheta)) - \lambda Z_2(0, \vartheta, z_1^0(\vartheta), z_2^0(\vartheta))$$
$$+ 2\lambda^2 z_1^0 \Theta(0, \vartheta, z_1^0(\vartheta), z_2^0(\vartheta)) \Big]. \tag{3.6}$$

The right-hand side of this nonhomogeneous equation is found to be primes denote the total derivative with respect to ϑ)

$$Z_1' + \lambda z_2^0 \Theta' - \lambda Z_2 + 2\lambda^2 z_1^0 \Theta$$
$$= -\frac{1}{16}(2\lambda^2 H + 26\lambda^2 + 32\lambda - 3H + 1) z_1^0 \sin\vartheta$$
$$+ \frac{1}{16}(6\lambda^2 H + 6\lambda^2 - 9H - 9) z_1^0 \sin 3\vartheta$$
$$+ \frac{1}{16}(5\lambda H - 15\lambda - 16) z_2^0 \cos\vartheta + \frac{3}{16}\lambda(1 + H) z_2^0 \cos 3\vartheta$$
$$+ \frac{1}{4}(-2\lambda^3 H + 3\lambda^2 + 4\lambda H - 4) z_1^0 \sin 2\vartheta$$
$$+ \frac{1}{4}\lambda^2(\lambda H - 2) z_2^{0^2} \sin 2\vartheta - \frac{1}{4}\lambda^2(9\lambda + 3H + 12) z_1^0 z_2^0$$
$$+ \frac{1}{4}(3\lambda^3 - 5\lambda^2 H - 4\lambda + 4H) z_1^0 z_2^0 \cos 2\vartheta$$
$$- \frac{1}{4}(2\lambda^2 H + 1) z_1^{0^3} \sin\vartheta + \frac{1}{4}\lambda(6\lambda^2 H + 6\lambda^2 - 4H - 5)$$
$$\times z_1^{0^2} z_2^0 \cos\vartheta + \frac{1}{4}(2\lambda^2 + H) z_1^0 z_2^{0^2} \sin\vartheta$$
$$- \frac{1}{4}\lambda(2\lambda^2 H + 2\lambda^2 - H) z_2^{0^3} \cos\vartheta. \tag{3.7}$$

The nonhomogeneous part of (3.6) involves trigonometric functions of ϑ with cyclic frequencies

$$\text{(a)}\ \frac{p-q}{p}, \quad \text{(b)}\ \frac{2(p-q)}{p}, \quad \text{(c)}\ \frac{|p-3q|}{p},$$

$$\text{(d)}\ \frac{p+q}{p}, \quad \frac{3p+q}{p}, \quad \frac{3p-q}{p}, \quad 2, \quad \frac{2(p+q)}{p}, \quad \frac{2q}{p}, \quad \frac{p+3q}{p}.$$

Let us determine when one of these frequencies coincides with the cyclic frequency q/p of the generating solution

$$\text{(a)}\ \frac{p-q}{p}=\frac{q}{p},\quad q=1,\quad p=2,\quad \lambda=\frac{1}{2};$$

$$\text{(b)}\ \frac{2(p-q)}{p}=\frac{q}{p},\quad q=2,\quad p=3,\quad \lambda=\frac{2}{3};$$

$$\text{(c)}\ \frac{p-3q}{p}=\frac{q}{p},\quad q=1,\quad p=4,\quad \lambda=\frac{1}{4};$$

$$\frac{3q-p}{p}=\frac{q}{p},\quad q=1,\quad p=2,\quad \lambda=\frac{1}{2}.$$

Such a coincidence is impossible in case (d). In cases (a), (b), and (c), equation (3.6) has $2p\pi$-periodic solutions for q and p given above only for those values of C and D for which the terms with $\sin(q\vartheta/p)$ and $\cos(q\vartheta/p)$ in equation (3.6) vanish. The equations for "generating amplitudes" when $\lambda=\frac{1}{4}, \frac{1}{2}, \frac{2}{3}$, however, yield only zero solutions: $C=D=0$. This means that no periodic solution exists for these values of λ. For all other values of λ given by formula (3.5), periodic solutions exist for any C and D. This means that for all rational $\lambda \in (0, 1)$ except the resonances $\lambda=\frac{1}{4}, \frac{1}{2}, \frac{2}{3}$ the periodic solution is general with four arbitrary constants C, D, μ, and t_0. By virtue of (2.1), (2.2), (3.4), and (3.5), the indicated solution can be written as (see also Chapter III, Subsection 1.2)

$$\xi=\frac{\mu}{\sqrt{1+C^2+D^2}}\cos\vartheta+O(\mu^2),$$

$$\eta=\frac{\mu}{\sqrt{1+C^2+D^2}}\sin\vartheta+O(\mu^2),$$

$$\zeta=\frac{\mu}{\sqrt{1+C^2+D^2}}\left(C\cos\frac{q}{p}\vartheta+D\sin\frac{q}{p}\vartheta\right)+O(\mu^2),$$

$$\chi=\frac{\mu}{\sqrt{1+C^2+D^2}}\left(-D\cos\frac{q}{p}\vartheta+C\sin\frac{q}{p}\vartheta\right)+O(\mu^2),$$

$$\vartheta=(N+\omega)(t-t_0)+O(\mu);\quad \frac{q}{p}\neq\frac{1}{4}, \frac{1}{2}, \frac{2}{3}. \tag{3.8}$$

We recall that the transition to the initial variables is carried out by means of formulas (1.9), (1.5), and (1.2).

The quantitative description of motion is more easily realized, however, if the equations of motion are first reduced to normal form; this is the subject of the next subsection.

2.4. Reduction of equations of motion to diagonal form and transformation to normal form. Let us apply to system (1.3) the linear

transformation defined by matrix (1.8) (the inverse transformation is also given below)

$$
\begin{aligned}
&\alpha = x_{-1} + x_1 + x_{-2} + x_2 = 2\,\mathrm{Re}\,(x_1 + x_2),\\
&\mathrm{B} = i\,(-x_{-1} + x_1 - x_{-2} + x_2) = -2\,\mathrm{Im}\,(x_1 + x_2),\\
&\Gamma = -x_{-1} - x_1 + x_{-2} + x_2 = 2\,\mathrm{Re}\,(x_2 - x_1),\\
&\Delta = i\,(-x_{-1} + x_1 + x_{-2} - x_2) = 2\,\mathrm{Im}\,(x_2 - x_1);
\end{aligned}
\tag{4.1}
$$

$$
x_\nu = \frac{1}{4}\,[\alpha + (-1)^\nu\,\Gamma - i\mathrm{B}\,\mathrm{sign}\,\nu + i\,(-1)^\nu \Delta\,\mathrm{sign}\,\nu] \tag{4.2}
$$

$$
(\nu = \mp 1,\ \mp 2).
$$

Introduction of dimensionless time τ (see (1.11)) yields a system of equations of the type (1, 1.6)

$$
\begin{aligned}
\frac{dx_1}{d\tau} = {} & ix_1 - \frac{3}{8}(1+H)\,x_{-1}^2 + \frac{1}{8}(H-7)\,x_1^2\\
& - \frac{1}{8}(1+H)(2\lambda+1)\,x_{-2}^2 + \frac{1}{8}(1+H)(2\lambda-1)\,x_2^2\\
& + \frac{1}{4}(3+H)\,x_{-1}x_1 + \frac{1}{4}(H-1)(2+\lambda)\,x_{-1}x_{-2}\\
& - \frac{1}{4}(1+H)(2-\lambda)\,x_{-1}x_2 + \frac{1}{4}\lambda\,(3-H)\,x_1x_{-2}\\
& - \frac{1}{4}\lambda\,(3+H)\,x_1x_2 + \frac{1}{4}(H-1)\,x_{-2}x_2 + (3),\\
\frac{dx_2}{d\tau} = {} & i\lambda x_2 + \frac{1}{8}(1+H)(2+\lambda)\,x_{-1}^2 - \frac{1}{8}(1+H)(2-\lambda)\,x_1^2\\
& + \frac{3}{8}\lambda\,(1+H)\,x_{-2}^2 + \frac{1}{8}\lambda\,(7-H)\,x_2^2 + \frac{1}{4}\lambda\,(1-H)\,x_{-1}x_1\\
& + \frac{1}{4}(1-H)(2\lambda+1)\,x_{-1}x_{-2} + \frac{1}{4}(H-3)\,x_{-1}x_2\\
& + \frac{1}{4}(H+1)(2\lambda-1)\,x_1x_{-2} + \frac{1}{4}(3+H)\,x_1x_2\\
& - \frac{1}{4}\lambda\,(3+H)\,x_{-2}x_2 + (3).
\end{aligned}
\tag{4.3}
$$

Here the third-power terms are not given explicitly because not all of them will be required later; the first and third equations are not given either since $x_{-1} = \overline{x_1}$ and $x_{-2} = \overline{x_2}$ (see (4.2)).

We define α_{lm}^{ν} by formulas (1, 3.2). Obviously $\alpha_{-l-m}^{-\nu} = \overline{\alpha_{lm}^{\nu}}$ (ν, l, $m = \mp 1, \mp 2$); only one of the tetrad α_{lm}^{ν}, α_{ml}^{ν}, $\alpha_{-l-m}^{-\nu}$, and

$\alpha^{-\nu}_{-m-}$ (or of the pair α^{ν}_{ll} and $\alpha^{-\nu}_{-l-l}$) is given below

$$\alpha^{-1}_{-1\,-1} = \overline{\alpha^{-2}_{-2\,-2}} = \frac{1}{8}\, i\,(H-7),$$

$$\alpha^{-1}_{11} = \alpha^{-1}_{-2\,-2} = \alpha^{-1}_{22} = \alpha^{-1}_{1\,-2} = \overline{\alpha^{-2}_{-1\,-1}} = \overline{\alpha^{-2}_{11}} = \overline{\alpha^{-2}_{22}} = \overline{\alpha^{-2}_{-1\,2}} = \frac{1}{8}\, i\,(1+H),$$

$$\alpha^{-1}_{-1\,1} = \alpha^{-1}_{-1\,-2} = \overline{\alpha^{-2}_{-1\,-2}} = \overline{\alpha^{-2}_{-2\,2}} = -\frac{1}{8}\, i\,(3+H),$$

$$\alpha^{-1}_{-1\,2} = \overline{\alpha^{-2}_{1\,-2}} = \frac{1}{8}\, i\,(H-3),$$

$$\alpha^{-1}_{12} = \alpha^{-1}_{-2\,2} = \overline{\alpha^{-2}_{-1\,1}} = \overline{\alpha^{-2}_{12}} = \frac{1}{8}\, i\,(1-H). \tag{4.4}$$

For $\lambda = \frac{1}{2}$, we select the same values for $\alpha^{-1}_{-2\,-2}$ and $\alpha^{-2}_{-1\,2}$.

Formulas (1, 3.3) yield

$$f_{-1} = f_1 = f_{-2} = f_2 = 0.$$

This means (see the end of Subsection 1.2) that resonant terms are eliminated when $\lambda = \frac{1}{2}$, and that, within second-power terms, the normal form

$$\frac{dy_\nu}{d\tau} = \lambda_\nu y_\nu \tag{4.5}$$

$$(\lambda_1 = \overline{\lambda_{-1}} = i, \quad \lambda_2 = \overline{\lambda_{-2}} = \lambda i; \quad \nu = \mp 1,\ \mp 2)$$

is valid for the case in question for $\lambda = \frac{1}{2}$ as well, that is, for all $\lambda \in (0, 1)$ without even this single possible exception.

2.5. General solution of the Cauchy problem. The general solution of system (4.5)

$$y_\nu = e^{\lambda_\nu \tau} y_\nu(0) \quad (\nu = \mp 1,\ \mp 2)$$

urnishes the general solution of the Cauchy problem for the initial system (1.1) within the assumed degree of approximation, this is achieved by means of substitutions (1, 2.3), (4.1), and (1.5). For instance, we have for α

$$\alpha = 2\,\mathrm{Re}\,(x_1 + x_2) = 2\,\mathrm{Re}\,[y_1 + y_2 + \sum (\alpha^1_{lm} + \alpha^2_{lm} y)\,{}_l y_m]$$
$$= 2\,\mathrm{Re}\,[e^{i\tau} y_1(0) + e^{i\lambda\tau} y_2(0) + \sum (\alpha^1_{lm} + \alpha^2_{lm})\, e^{(\lambda_l + \lambda_m)\tau} y_l(0)\, y_m(0)],$$

whence

$$\frac{1}{2}\alpha = \operatorname{Re} y_1(0)\cos\tau - \operatorname{Im} y_1(0)\sin\tau + \operatorname{Re} y_2(0)\cos\lambda\tau$$

$$- \operatorname{Im} y_2(0)\sin\lambda\tau - \sum_{l,\, m=\mp 1,\, \mp 2} \frac{1}{i}(\alpha^1_{lm} + \alpha^2_{lm})$$

$$\times \Big\{[\operatorname{Re} y_l(0)\operatorname{Im} y_m(0) + \operatorname{Re} y_m(0)\operatorname{Im} y_l(0)]\cos\frac{1}{i}(\lambda_l + \lambda_m)\tau$$

$$+ [\operatorname{Re} y_l(0)\operatorname{Re} y_m(0) - \operatorname{Im} y_l(0)\operatorname{Im} y_m(0)]\sin\frac{1}{i}(\lambda_l + \lambda_m)\tau\Big\}. \quad (5.1)$$

The constants $y_\nu(0)$ now have to be expressed in terms of the initial values of the variables, i.e. α_0, β_0, γ_0, and Δ_0. Inversion of (1, 2.3) yields

$$y_\nu(0) = x_\nu(0) - \sum \alpha^\nu_{jh} x_j(0) x_h(0) + (3) \quad (\nu = \mp 1,\ \mp 2),$$

after which (4.2) and (1.5) give

$$y_\nu(0) = \frac{1}{4}\left\{\alpha_0 + (-1)^\nu \frac{N}{V}\gamma_0 + i \operatorname{sign}\nu\left[-\frac{N}{V}\beta_0 + (-1)^\nu\Delta_0\right]\right\}$$

$$- \frac{1}{16}\sum \alpha^\nu_{jh}\left\{\alpha_0 + (-1)^j\frac{N}{V}\gamma_0 + i\operatorname{sign} j\left[-\frac{N}{V}\beta_0 + (-1)^j\Delta_0\right]\right\}$$

$$\times\left\{\alpha_0 + (-1)^h\frac{N}{V}\gamma_0 + i\operatorname{sign} h\left[-\frac{N}{V}\beta_0 + (-1)^h\Delta_0\right]\right\}.$$

It then follows that

$$\operatorname{Re} y_\nu(0) = \frac{1}{4}\left[\alpha_0 + (-1)^\nu\frac{N}{V}\gamma_0\right]$$

$$+ \frac{1}{16}\sum_{,\, h=\mp 1,\, \mp 2}\frac{1}{i}\alpha^\nu_{jh}\left\{\operatorname{sign} h\left[\alpha_0 + (-1)^j\frac{N}{V}\gamma_0\right]\right.$$

$$\times\left[-\frac{N}{V}\beta_0 + (-1)^h\Delta_0\right] + \operatorname{sign} j\left[\alpha_0 + (-1)^h\frac{N}{V}\gamma_0\right]$$

$$\left.\times\left[-\frac{N}{V}\beta_0 + (-1)^j\Delta_0\right]\right\},$$

$$\operatorname{Im} y_\nu(0) = \frac{1}{4}\left[-\frac{N}{V}\beta_0 + (-1)^\nu\Delta_0\right]$$

$$- \frac{1}{16}\sum_{j,\, h=\mp 1,\, \mp 2}\frac{1}{i}\alpha^\nu_{jh}\left\{\left[\alpha_0 + (-1)^j\frac{N}{V}\gamma_0\right]\left[\alpha_0 + (-1)^h\frac{N}{V}\gamma_0\right]\right.$$

$$\left.- \operatorname{sign}(jh)\left[\frac{N}{V}\beta_0 - (-1)^j\Delta_0\right]\left[\frac{N}{V}\beta_0 - (-1)^h\Delta_0\right]\right\} \quad (5.2)$$

$$(\nu = \mp 1,\ \mp 2).$$

Summation indices take on the values ∓ 1, ∓ 2. Formulas for β, γ, and Δ (and ε; see (1.2)) can be written similarly to (5.1). Note that

β and γ, as compared to α and Δ, are of the order of V/N (a quantity whose square is negligible in comparison to unity); in fact, this result can be obtained from the linear approximation.

Formulas (5.1), (5.2), and (4.4) thus represent the solution of system (1.1) as almost periodic (for irrational λ) or periodic (for rational λ).

2.6. Preliminary conclusions on stability. Until this subsection the terms considered in normal forms were of power not higher than two; as follows from the general theory and as was demonstrated in Subsection 2.5, in case of two pairs of pure imaginary roots these terms do not violate the neutrality of the approximation. Now we shall analyze the effect of third-power terms. The coefficients g_1, g_2, h_1, and h_2 of system (1, 2.10) are found from formulas (1, 3.5)

$$\begin{aligned}
g_1 &= 3b^1_{11-1} + 2\,[2\,(a^1_{1-1}\alpha^{-1}_{1-1} + a^1_{11}\alpha^1_{1-1} + a^1_{1-2}\alpha^{-2}_{1-1} + a^1_{12}\alpha^2_{1-1}) \\
&\qquad + a^1_{-1-1}\alpha^{-1}_{11} + a^1_{-11}\alpha^1_{11} + a^1_{-1-2}\alpha^{-2}_{11} + a^1_{-12}\alpha^2_{11}], \\
g_2 &= 3b^2_{22-2} + 2\,[2\,(a^2_{2-1}\alpha^{-1}_{2-2} + a^2_{21}\alpha^1_{2-2} + a^2_{2-2}\alpha^{-2}_{2-2} + a^2_{22}\alpha^2_{2-2}) \\
&\qquad + a^2_{-2-1}\alpha^{-1}_{22} + a^2_{-21}\alpha^1_{22} + a^2_{-2-2}\alpha^{-2}_{22} + a^2_{-22}\alpha^2_{22}], \\
h_1 &= 6b^1_{12-2} + 2\,(a^1_{1-1}\alpha^{-1}_{2-2} + a^1_{11}\alpha^1_{2-2} + a^1_{1-2}\alpha^{-2}_{2-2} + a^1_{12}\alpha^2_{2-2} \\
&\qquad + a^1_{2-1}\alpha^{-1}_{1-2} + a^1_{21}\alpha^1_{1-2} + a^1_{2-2}\alpha^{-2}_{1-2} + a^1_{22}\alpha^2_{1-2} \\
&\qquad + a^1_{-2-1}\alpha^{-1}_{12} + a^1_{-21}\alpha^1_{12} + a^1_{-2-2}\alpha^{-2}_{12} + a^1_{-22}\alpha^2_{12}), \\
h_2 &= 6b^2_{21-1} + 2\,(a^2_{2-1}\alpha^{-1}_{1-1} + a^2_{21}\alpha^1_{1-1} + a^2_{2-2}\alpha^{-2}_{1-1} + a^2_{22}\alpha^2_{1-1} \\
&\qquad + a^2_{1-1}\alpha^{-1}_{2-1} + a^2_{11}\alpha^1_{2-1} + a^2_{1-2}\alpha^{-2}_{2-1} + a^2_{12}\alpha^2_{2-1} \\
&\qquad + a^2_{-1-1}\alpha^{-1}_{21} + a^2_{-11}\alpha^1_{21} + a^2_{-1-2}\alpha^{-2}_{21} + a^2_{-12}\alpha^2_{21}).
\end{aligned}$$

The coefficients α^ν_{lm} are given by formulas (4.4); a^ν_{jh} are the coefficients of the corresponding quadratic terms in equations (4.3); and b^ν_{jhk} are the coefficients of the corresponding third-power terms written as (3) in equations (4.3). These last coefficients can be derived from system (1.3) by means of transformations (4.1), (4.2), and (1.11). We emphasize that, prior to the computation, equations (4.3) must be reduced to the form of (1, 2.1), that is, (1, 2.2) must be satisfied. Therefore,

$$\begin{aligned}
\frac{3}{i}\,b^1_{11-1} &= \frac{21}{16} + \frac{11}{16}\,H + \frac{\lambda}{16}\,(5 - 5H + 4H^2), \\
\frac{3}{i}\,b^2_{22-2} &= \frac{5}{16} - \frac{7}{16}\,H + \frac{\lambda}{16}\,(21 + 13H + 4H^2), \\
\frac{6}{i}\,b^1_{12-2} &= -\frac{3}{8} + \frac{3}{8}\,H + \frac{\lambda}{8}\,(-11 - 5H + 4H^2), \\
\frac{6}{i}\,b^2_{21-1} &= -\frac{11}{8} - \frac{7}{8}\,H + \frac{\lambda}{8}\,(-3 + 5H + 4H^2),
\end{aligned}$$

$$g_1 = \frac{1}{8} iH(1+2H)(\lambda-1), \quad g_2 = 0,$$

$$h_1 = \frac{1}{4} iH(1+2H)(\lambda-1), \quad h_2 = 0.$$

Since the quantities g_1 and g_2 are pure imaginary, the Molchanov criterion (Subsection 1.4) fails. Note that, by equations (1, 4.1), in this case the expressions

$$|y_1|^2 = c_1 \quad \text{and} \quad |y_2|^2 = c_2$$

are the first integrals of system (1, 2.10) and can be used to construct the Lyapunov function of this system.

The determinant J (see (1, 5.2)) vanishes, so that the Bibikov-Pliss criterion (Subsection 1.5) fails as well.

2.7. Construction of the Lyapunov function. By using (1.11) to substitute the independent variable τ into system (1.3), we obtain

$$\begin{aligned}
\frac{d\alpha}{d\tau} &= \frac{1}{2}(1+\lambda)\mathrm{B} + \frac{1}{2}(1-\lambda)\Delta - \frac{1}{4}(1-\lambda)\alpha^2 \\
&\quad + \frac{1}{2}(1-\lambda)\left(1+\frac{1}{2}H\right)\Delta^2 + \frac{1}{2}(1+\lambda)\mathrm{B}\Delta - \frac{1}{4}(1-\lambda)\alpha^2\Delta \\
&\quad + \frac{1}{2}(1-\lambda)\left(1+\frac{5}{6}H\right)\Delta^3 + \frac{1}{2}(1+\lambda)\left(1+\frac{1}{2}H\right)\mathrm{B}\Delta^2, \\
\frac{d\mathrm{B}}{d\tau} &= -\frac{1}{2}(1+\lambda)\alpha + \frac{1}{2}(1-\lambda)\Gamma + \frac{1}{2}(1+\lambda)\mathrm{B}\Gamma \\
&\quad + \frac{1}{2}(1-\lambda)\Gamma\Delta + \frac{1}{12}(1+\lambda)\alpha^3 - \frac{1}{4}(1-\lambda)\alpha^2\Gamma \\
&\quad + \frac{1}{2}(1-\lambda)\left(1+\frac{1}{2}H\right)\Gamma\Delta^2 + \frac{1}{2}(1+\lambda)\mathrm{B}\Gamma\Delta, \\
\frac{d\Gamma}{d\tau} &= -\frac{1}{2}(1-\lambda)\mathrm{B} - \frac{1}{2}(1+\lambda)\Delta + \frac{1}{4}(1+\lambda)\alpha^2 \\
&\quad - \frac{1}{2}(1+\lambda)\mathrm{B}^2 - \frac{1}{4}(1+\lambda)H\Delta^2 - \frac{1}{2}(1-\lambda)\mathrm{B}\Delta \\
&\quad + \frac{1}{12}(1+\lambda)H\Delta^3 + \frac{1}{4}(1-\lambda)\alpha^2\mathrm{B} \\
&\quad - \frac{1}{2}(1-\lambda)\left(1+\frac{1}{2}H\right)\mathrm{B}\Delta^2 - \frac{1}{2}(1+\lambda)\mathrm{B}^2\Delta, \\
\frac{d\Delta}{d\tau} &= -\frac{1}{2}(1-\lambda)\alpha + \frac{1}{2}(1+\lambda)\Gamma + \frac{1}{2}(1-\lambda)H\alpha\Delta \\
&\quad - \frac{1}{2}(1+\lambda)H\Gamma\Delta - \frac{1}{2}(1-\lambda)\frac{V^2}{N^2}\mathrm{B}\Gamma + \frac{1}{12}(1-\lambda)\alpha^3 \\
&\quad - \frac{1}{2}(1-\lambda)H\left(\frac{1}{2}+H\right)\alpha\Delta^2 + \frac{1}{2}(1+\lambda)H\left(\frac{1}{2}+H\right)\Gamma\Delta^2
\end{aligned} \tag{7.1}$$

within the third-power terms. The derivative of the function

$$W=\alpha^2\left(1-\Delta-\frac{1}{3}\alpha^2\right)+\mathrm{B}^2+\Gamma^2 + \Delta^2\left(1+H\Delta-\frac{1}{3}H\Delta^2\right)+\frac{1}{4}(\alpha^2-H\Delta^2)^2, \quad (7.2)$$

derived in accordance with equations (7.1), is zero. If

$$\Delta<1-\frac{1}{3}\alpha^2, \quad \Delta_1<\Delta<\Delta_2, \quad (7.3)$$

where Δ_1 and Δ_2 are the roots of the quadratic equation,

$$H\Delta^2-3H\Delta-3=0,$$

that is,

$$\Delta_{1,2}=\frac{3}{2}\left(1\mp\sqrt{1+\frac{4}{3}\frac{1}{H}}\right),$$

then the function W is positive-definite in the sense of Lyapunov. Note that

$$-\frac{1}{H}<\Delta_1<0, \quad 3<\Delta_2.$$

The trivial solution of system (7.1) is therefore stable in the sense of Lyapunov, and domain (7.3) in the α-, β-, γ-, and Δ-space is the domain of the allowed initial conditions.

Both systems (7.1) and (1.3) are, however, approximations of system (1.1), although the order of this approximation is quite high: within the third-power terms. Nevertheless, we are justified in stating only the formal stability [238e] of the equilibrium position $\alpha=\beta=\gamma=0$, $\varepsilon=\varepsilon_0$ of system (1.1) until we establish convergence of the first integral of system (1.1); this integral must be represented in expansion (7.2) within the fourth-power terms.

§ 3. The Trajectory Described by the Centre of a Shaft's Cross Section in One Revolution

3.1. Statement of the problem and equations of motion. A large number of studies have been published on both the theoretical and experimental aspects of oscillations in rotor systems. Many scientists share the opinion that most of the important problems in this field are completely solved. Nevertheless, a number of the phenomena observed in rotor systems have received little attention. For instance, no explanation has been suggested for fatigue failure in rotors working in a steady-state mode (EVA-type spindles). Indeed, in terms of accepted notions, any cross section along the shaft length, given steady-state rotation in equirigid or absolutely rigid supports, de-

scribes a circular trajectory. Hence, the shaft undergoes only static stresses (with the exception of its own weight), which cannot lead to fatigue failure. Experience demonstrates, however, that a circular trajectory is a very rare phenomenon in rotors. Noncircular trajectories are generated as a result of noncircular orifices in the outer races of the bearings. This imperfection exists in all supports of the bearings and its origin lies in the pulsing of borers and reamers. It is thus clear that an analysis of the shape of trajectories described by the centre of a shaft's cross section in one revolution should be of both practical and theoretical interest. To the knowledge of the author, no such analysis was carried out until Popov's publications appeared [345, 346a, b].

The choice of a simplified model that simulates the qualitative behaviour of the system with sufficient accuracy is an especially complicated problem. Let us choose as the simplest model a weightless vertical shaft (Fig. 15) with mass m mounted on this shaft with eccentricity e. The mass is set equal to the reduced mass calculated by the standard method [50], [51]. We assume that the shaft, being absolutely rigid for torsion, is mounted in absolutely rigid bearings and driven at a constant angular speed ω_0 by a driving member rigidly coupled to the shaft. Let the mass deviate from the equilibrium trajectory at a chosen moment in time. Then the speed of motion along the trajectory is not constant. The inertial force applied to the mass becomes noncentral and generates, in addition to the radial force, a tangential elastic force. Indeed, were it not for lateral bending of the shaft, it would be impossible for the mass to move along the trajectory in the direction of rotation during the acceleration period. The system's potential energy can therefore be expressed as the sum of the works of inertial forces over the paths of radial and tangential strains, that is,

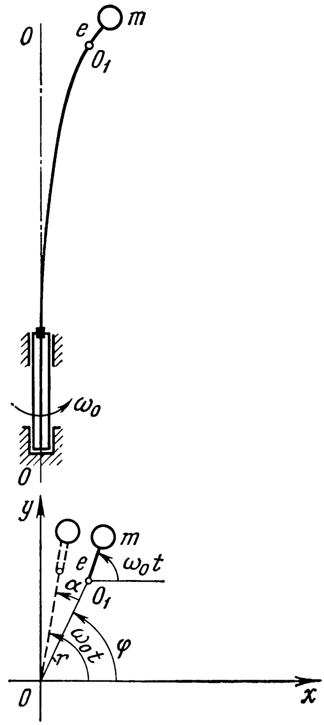

FIG. 15

$$\Pi = \frac{1}{2} kr^2 (1 + \alpha^2), \qquad (1.1)$$

where α is the deviation of the radius-vector r of the point where the mass is fixed to the shaft from the position calculated for $\dot{\varphi} = \omega_0 = \text{const}$.

We now calculate the kinetic energy of the system, shifting to polar coordinates according to the formulas

$$x_s = r\cos(\omega_0 t - \alpha) + e\cos\omega_0 t,$$
$$y_s = r\sin(\omega_0 t - \alpha) + e\sin\omega_0 t,$$

$$K = \frac{1}{2} m \left\{ \left(\frac{dr}{dt}\right)^2 + r^2\left(\omega_0 - \frac{d\alpha}{dt}\right)^2 + \omega_0 e \operatorname{sign} c \left[-2\frac{dr}{dt}\sin\alpha + 2r\left(\omega_0 - \frac{d\alpha}{dt}\right)^2 \cos\alpha + \omega_0 \right] \right\}, \quad (1.2)$$

where k is the shaft stiffness at the juncture of the shaft and the mass, $c = p^2 - \omega_0^2$, and p is the natural cyclic frequency of flexural vibrations of the shaft.

The equations of motion of the mass-carrying point of the shaft are

$$\frac{d^2 r}{dt^2} - r\left(\omega_0 - \frac{d\alpha}{dt}\right)^2 - \omega_0^2 e\cos\alpha \operatorname{sign} c + p^2 r(1 + \alpha^2) = 0,$$
$$r^2 \frac{d^2\alpha}{dt^2} - 2r\left(\omega_0 - \frac{d\alpha}{dt}\right)\frac{dr}{dt} + \omega_0^2 e \sin\alpha \operatorname{sign} c + p^2 r^2 \alpha = 0. \quad (1.3)$$

Assuming angle α sufficiently small and replacing $\cos\alpha$ by the first two terms of its expansion, we obtain

$$\frac{d^2 r}{dt^2} + cr + 2\omega_0 r\frac{d\alpha}{dt} - r\left(\frac{d\alpha}{dt}\right)^2 + p^2 r\alpha^2 + \frac{1}{2}\omega_0^2 e\alpha^2 \operatorname{sign} c - \omega_0^2 e \operatorname{sign} c = 0,$$
$$\frac{d^2\alpha}{dt^2} + 2\frac{1}{r}\frac{dr}{dt}\frac{d\alpha}{dt} - 2\omega_0\frac{1}{r}\frac{dr}{dt} + p^2\alpha + \omega_0^2 e\frac{\alpha}{r}\operatorname{sign} c = 0. \quad (1.4)$$

The first equation includes a free term $\omega_0^2 e$; it would be desirable to eliminate this term in order to obtain a circular path for the zero solution. Note that the unperturbed motion with

$$r_0 = \left| \frac{\omega_0^2 e}{c} \right|$$

is obtained from (1.3) for $\alpha = 0$. Introduction of new variables

$$z_{-1} = r - r_0, \quad z_1 = \frac{dz_{-1}}{dt}, \quad z_{-2} = \alpha, \quad z_2 = \frac{dz_{-2}}{dt} \quad (1.5)$$

transforms the equations of perturbed motion to

$$\frac{dz_1}{dt}+cz_{-1}+2\omega_0 r_0 z_2=r_0 z_2^2+z_{-1}z_2^2-2\omega_0 z_{-1}z_2 -\left(p^2 r_0+\frac{1}{2}\,\omega_0^2 e\operatorname{sign} c\right) z_{-2}^2-p^2 z_{-1}z_{-2}^2,$$

$$\frac{dz_2}{dt}+2\,\frac{1}{r_0+z_{-1}}\,z_1 z_2-2\omega_0\,\frac{1}{r_0+z_{-1}}\,z_1 +p^2 z_{-2}+\omega_0^2 e\,\frac{1}{r_0+z_{-1}}\,z_{-2}\operatorname{sign} c=0. \qquad (1.6)$$

Assuming that the amplitude of the perturbed motion is smaller than that of the unperturbed motion, that is, $|z_{-1}|<r_0$, we express the fraction

$$\frac{1}{r_0+z_{-1}}$$

as a convergent binomial series and truncate it to the first three terms of the expansion

$$r_0^{-1}\left(1+\frac{z_{-1}}{r_0}\right)^{-1}=r_0^{-1}\left(1-\frac{z_{-1}}{r_0}+\frac{z_{-1}^2}{r_0^2}\right).$$

Substituting this expansion into (1.6) and taking into account (1.5), we arrive at the autonomous fourth-order system

$$\frac{dz_{-1}}{dt}=z_1,$$

$$\frac{dz_1}{dt}=-cz_{-1}-2\omega_0 r_0 z_2+f_1(z_{-1},\ z_1,\ z_{-2},\ z_2),$$

$$\frac{dz_{-2}}{dt}=z_2,$$

$$\frac{dz_2}{dt}=2\,\frac{\omega_0}{r_0}\,z_1-(2p^2-\omega_0^2)\,z_{-2}+f_2(z_{-1},\ z_1,\ z_{-2},\ z_2), \qquad (1.7)$$

where

$$f_1=-2\omega_0 z_{-1}z_2-\left(p^2 r_0+\frac{1}{2}\,\omega_0^2 e\operatorname{sign} c\right) z_{-2}^2+r_0 z_2^2-p^2 z_{-1}z_{-2}^2+z_{-1}z_{-2}^2,$$

$$f_2=-2\frac{\omega_0}{r_0^2}\,z_{-1}z_1+\frac{\omega_0^2 e}{r_0^2}\,z_{-1}z_{-2}-\frac{2}{r_0}\,z_1 z_2 +2\frac{\omega_0}{r_0^3}\,z_{-1}^2 z_1-\frac{\omega_0^2 e}{r_0^3}\,z_{-1}^2 z_{-2}+\frac{2}{r_0^2}\,z_{-1}z_1 z_2. \qquad (1.8)$$

In vector notation, we obtain

$$\frac{d\mathbf{z}}{dt}=\mathbf{Az}+\mathbf{f}(\mathbf{z}), \qquad (1.7a)$$

where $\mathbf{z}=(z_{-1},\ z_1,\ z_{-2},\ z_2)^{\tau}$, $\mathbf{A}$ is a square matrix composed of coefficients of the linear part of system (1.7), and $\mathbf{f}(\mathbf{z})=(0,\ f_1,\ 0,\ f_2)^{\tau}$.

3.2. Reduction to diagonal form. The eigenvalues of the matrix **A** in (1.7) are $\mp\omega_1 i$ and $\mp\omega_2 i$, where

$$\omega_{1,2} = +\frac{\sqrt{2}}{2}\sqrt{3p^2+2\omega_0^2 \mp p\sqrt{p^2+24\omega_0^2}}. \tag{2.1}$$

These eigenvalues are pure imaginary and distinct ($0 < \omega_1 < \omega_2$) if either of the following two inequalities is satisfied

$$\omega_0 < p, \quad \sqrt{2}p < \omega_0. \tag{2.2}$$

This is in agreement with the recommendation for selecting the range of effective angular velocities in the out-of-resonant region

$$\omega_{\text{eff}} > 1.4\omega_{\text{crit}},$$

which resulted from many years of experience with rotor systems. We assume henceforth that one of the conditions in (2.2) is satisfied.

We introduce dimensionless time

$$\tau = \omega_2 t$$

and rewrite the vector equation (1.7) as

$$\frac{d\mathbf{z}}{d\tau} = \frac{1}{\omega_2}\mathbf{A}\mathbf{z} + \frac{1}{\omega_2}\mathbf{f}(\mathbf{z}). \tag{2.3}$$

The eigenvalues of the matrix $\frac{1}{\omega_2}\mathbf{A}$ are

$$\lambda_{-1} = -i, \quad \lambda_1 = i, \quad \lambda_{-2} = -\lambda i, \quad \lambda_2 = \lambda i \quad \left(\lambda = \frac{\omega_1}{\omega_2} < 1\right). \tag{2.4}$$

The linear change of variables

$$\mathbf{z} = \mathbf{S}\mathbf{x}, \tag{2.5}$$

where **S** is a matrix comprising eigenvectors of the matrix $\frac{1}{\omega_2}\mathbf{A}$, reduces system (2.3) to diagonal form

$$\frac{d\mathbf{x}}{d\tau} = \operatorname{diag}(-i, \ i, -\lambda i, \ \lambda i)\,\mathbf{x} + \frac{1}{\omega_2}\mathbf{S}^{-1}\mathbf{f}(\mathbf{S}\mathbf{x}). \tag{2.6}$$

We compute **S** and $\mathbf{S}^{-1}$

$$\mathbf{S} = \begin{Vmatrix} 1 & 1 & 1 & 1 \\ -i\omega_2 & i\omega_2 & -i\omega_1 & i\omega_1 \\ id_2 & -id_2 & id_1 & -id_1 \\ d_2\omega_2 & d_2\omega_2 & d_1\omega_1 & d_1\omega_1 \end{Vmatrix},$$

$$\mathbf{S}^{-1} = \begin{Vmatrix} -D & iD_1 & iD_2 & D_3 \\ -D & -iD_1 & -iD_2 & D_3 \\ \frac{d_2}{d_1\lambda}D & -i\frac{d_2}{d_1}D_1 & -\frac{i}{\lambda}D_2 & -D_3 \\ \frac{d_2}{d_1\lambda}D & i\frac{d_2}{d_1}D_1 & \frac{i}{\lambda}D_2 & -D_3 \end{Vmatrix},$$

where the notation is

$$d_1 = \frac{\omega_1^2 - c}{2r_0\omega_0\omega_1}, \qquad d_2 = \frac{\omega_2^2 - c}{2r_0\omega_0\omega_2}, \qquad D = \frac{d_1\lambda}{2(d_2 - d_1\lambda)},$$

$$D_1 = \frac{d_1}{2\omega_2(d_1 - d_2\lambda)}, \qquad D_2 = \frac{\lambda}{2(d_1 - d_2\lambda)}, \qquad D_3 = \frac{1}{2\omega_2(d_2 - d_1\lambda)}. \tag{2.7}$$

Transformation (2.5) and its inverse are, in detailed form,

$$\begin{aligned}
z_{-1} &= x_{-1} + x_1 + x_{-2} + x_2,\\
z_1 &= -i\omega_2 x_{-1} + i\omega_2 x_1 - i\omega_1 x_{-2} + i\omega_1 x_2,\\
z_{-2} &= id_2 x_{-1} - id_2 x_1 + id_1 x_{-2} - id_1 x_2,\\
z_2 &= d_2\omega_2 x_{-1} + d_2\omega_2 x_1 + d_1\omega_1 x_{-2} + d_1\omega_1 x_2;
\end{aligned} \tag{2.8}$$

$$\begin{aligned}
x_{-1} &= -Dz_{-1} + iD_1 z_1 + iD_2 z_{-2} + D_3 z_2,\\
x_1 &= -Dz_{-1} - iD_1 z_1 - iD_2 z_{-2} + D_3 z_2,\\
x_{-2} &= \frac{d_2}{d_1\lambda} D_3 z_{-1} - i\frac{d_2}{d_1} D_1 z_1 - \frac{i}{\lambda} D_2 z_{-2} - D_3 z_2,\\
x_2 &= \frac{d_2}{d_1\lambda} D_3 z_{-1} + i\frac{d_2}{d_1} D_1 z_1 + \frac{i}{\lambda} D_2 z_{-2} - D_3 z_2.
\end{aligned} \tag{2.9}$$

In a more compact form, (2.9) can be written as

$$x_\nu = (-1)^\nu \left(\frac{d_2}{d_1\lambda}\right)^{|\nu|-1} Dz_{-1} + \operatorname{sign}\nu\,(-1)^\nu i \left(\frac{d_2}{d_1}\right)^{|\nu|-1} D_1 z_1$$

$$+ \operatorname{sign}\nu\,(-1)^\nu i \left(\frac{1}{\lambda}\right)^{|\nu|-1} D_2 z_{-2} - (-1)^\nu D_3 z_2 \quad (\nu = \mp 1, \mp 2). \tag{2.10}$$

Obviously, "diagonal" variables are complex conjugate: $x_{-1} = \bar{x}_1$ and $x_{-2} = \bar{x}_2$; consequently, transformation (2.8) can be represented by

$$\begin{aligned}
z_{-1} &= 2\,\mathrm{Re}\,(x_1 + x_2), & z_1 &= -2\,\mathrm{Im}\,(\omega_2 x_1 + \omega_1 x_2),\\
z_{-2} &= 2\,\mathrm{Im}\,(d_2 x_1 + d_1 x_2), & z_2 &= 2\,\mathrm{Re}\,(d_2\omega_2 x_1 + d_1\omega_1 x_2).
\end{aligned} \tag{2.11}$$

It is now possible to calculate components of the vector

$$\mathbf{h}(\mathbf{x}) = \frac{1}{\omega_2}\mathbf{S}^{-1}\mathbf{f}(\mathbf{S}\mathbf{x})$$

of the nonlinear part of system (2.6)

$$h_{-1}(x) = \frac{1}{\omega_2}[iD_1 f_1(\mathbf{S}\mathbf{x}) + D_3 f_2(\mathbf{S}\mathbf{x})], \quad h_1 = \frac{1}{\omega_2}[-iD_1 f_1 + D_3 f_2],$$

$$h_{-2} = \frac{1}{\omega_2}\left[-i\frac{d_2}{d_1}D_1 f_1 - D_3 f_2\right], \quad h_2 = \frac{1}{\omega_2}\left[i\frac{d_2}{d_1}D_1 f_1 - D_3 f_2\right]. \tag{2.12}$$

According to (1.8) and (2.8)

$$f_1(\mathbf{S}x) = \omega_2 \{a(x_{-1}+x_1)^2 + b(x_{-2}+x_2)^2 + g(x_{-1}+x_1)(x_{-2}+x_2) + [h(x_{-1}-x_1) + j(x_{-2}-x_2)]^2\},$$
$$f_2(\mathbf{S}x) = i\omega_2 [l(x_{-1}^2 - x_1^2) + n(x_{-2}^2 - x_2^2) + q(x_{-1}-x_1)(x_{-2}+x_2) + s(x_{-1}+x_1)(x_{-2}-x_2)], \quad (2.13)$$

where

$$a = d_2(d_2\omega_2 r_0 - 2\omega_0), \quad b = \lambda d_1(d_1\omega_1 r_0 - 2\omega_0),$$
$$g = d_2(d_1\omega_1 r_0 - 2\omega_0) + \lambda d_1(d_2\omega_2 r_0 - 2\omega_0),$$
$$h = +\frac{d_2}{\sqrt{\omega_2}}\sqrt{\left(p^2+\frac{1}{2}c\right)r_0}, \quad j = +\frac{d_1}{\sqrt{\omega_2}}\sqrt{\left(p^2+\frac{1}{2}c\right)r_0},$$
$$l = \frac{1}{r_0^2}\left(d_2 r_0 \frac{c}{\omega_2} + 2d_2\omega_2 r_0 + 2\omega_0\right),$$
$$n = \frac{1}{r_0^2}\left(d_1 r_0 \frac{c}{\omega_2} + 2\lambda\, d_1\omega_1 r_0 + 2\lambda\omega_0\right),$$
$$q = \frac{1}{r_0^2}\left(d_2 r_0 \frac{c}{\omega_2} + 2d_1\omega_1 r_0 + 2\omega_0\right),$$
$$s = \frac{1}{r_0^2}\left(d_1 r_0 \frac{c}{\omega_2} + 2d_2\omega_1 r_0 + 2\lambda\omega_0\right). \quad (2.14)$$

System reduced to diagonal form (2.6), in the symmetrized form used in this book, is

$$\frac{dx_\nu}{d\tau} = \lambda_\nu x_\nu + \sum_{j,\,h=\mp 1,\,\mp 2} a_{jh}^\nu x_j x_h + (3) \quad (\nu = \mp 1, \ \mp 2) \quad (2.15)$$
$$(a_{hj}^\nu = a_{jh}^\nu; \ \nu, \ j, \ h = \mp 1, \ \mp 2).$$

By formulas (2.12) and (2.13), the coefficients of the quadratic terms are

$$a_{-1-1}^{-1} = -a_{11}^1 = i(a+h^2)D_1 + ilD_3,$$
$$a_{-1-1}^{1} = -a_{11}^{-1} = -i(a+h^2)D_1 + ilD_3,$$
$$a_{-1-1}^{-2} = -a_{11}^{2} = -i\frac{d_2}{d_1}(a+h^2)D_1 - ilD_3,$$
$$a_{-1-1}^{2} = -a_{11}^{-2} = i\frac{d_2}{d_1}(a+h^2)D_1 - ilD_3,$$
$$a_{-2-2}^{-1} = -a_{22}^{1} = i(b+j^2)D_1 + inD_3,$$
$$a_{-2-2}^{1} = -a_{22}^{-1} = -i(b+j^2)D_1 + inD_3,$$
$$a_{-2-2}^{-2} = -a_{22}^{2} = -i\frac{d_2}{d_1}(b+j^2)D_1 - inD_3,$$
$$a_{-2-2}^{2} = -a_{22}^{-2} = i\frac{d_2}{d_1}(b+j^2)D_1 - inD_3,$$

$$a^{-1}_{-11} = -a^{1}_{-11} = i\,(a+h^2)\,D_1,$$

$$a^{-2}_{-11} = -a^{2}_{-11} = -i\,\frac{d_2}{d_1}\,(a+h^2)\,D_1,$$

$$2a^{-1}_{-1-2} = -2a^{1}_{12} = i\,(g+2hj)\,D_1 + i\,(q+s)\,D_3,$$

$$2a^{1}_{-1-2} = -2a^{-1}_{12} = -i\,(g+2hj)\,D_1 + i\,(q+s)\,D_3,$$

$$2a^{-2}_{-1-2} = -2a^{2}_{12} = -i\,\frac{d_2}{d_1}\,(g+2hj)\,D_1 - i\,(q+s)\,D_3,$$

$$2a^{2}_{-1-2} = -2a^{-2}_{12} = i\,\frac{d_2}{d_1}\,(g+2hj)\,D_1 - i\,(q+s)\,D_3,$$

$$2a^{-1}_{-12} = -2a^{1}_{1-2} = i\,(g-2hj)\,D_1 + i\,(q-s)\,D_3,$$

$$2a^{1}_{-12} = -2a^{-1}_{1-2} = -i\,(g-2hj)\,D_1 + i\,(q-s)\,D_3,$$

$$2a^{-2}_{-12} = -2a^{2}_{1-2} = -i\,\frac{d_2}{d_1}\,(g-2hj)\,D_1 - i\,(q-s)\,D_3,$$

$$2a^{2}_{-12} = -2a^{-2}_{1-2} = i\,\frac{d_2}{d_1}\,(g-2hj)\,D_1 - i\,(q-s)\,D_3,$$

$$a^{-1}_{-22} = -a^{1}_{-22} = i\,(b-j^2)\,D_1,$$

$$a^{-2}_{-22} = -a^{2}_{-22} = -i\,\frac{d_2}{d_1}\,(b+j^2)\,D_1. \tag{2.16}$$

Consequently, all the coefficients of the quadratic terms (as well as the linear ones) are found to be pure imaginary $a^{-\nu}_{-j-h} = \overline{a^{\nu}_{jh}} = -a^{\nu}_{jh}$ $(\nu,\ j,\ h = \mp 1,\ \mp 2)$.

3.3 Reduction to normal form. We assume (see the beginning of Subsection 3.2) that

$$\omega_2 \neq 2\omega_1, \tag{3.1}$$

that is, we exclude the case $\lambda = \frac{1}{2}$ from the analysis. We have already demonstrated in Subsection 1.2 that, in this situation, normal form becomes, within the second-power terms,

$$\frac{dy_\nu}{d\tau} = \lambda_\nu y_\nu + (3) \quad (\nu = \mp 1,\ \mp 2). \tag{3.2}$$

The normalizing transformation then becomes

$$x_\nu = y_\nu + \sum_{l,\ m=\mp 1,\ \mp 2} \alpha^{\nu}_{lm} y_l y_m + (3) \quad (\nu = \mp 1,\ \mp 2), \tag{3.3}$$

where the coefficients are given by formulas (1, 3.1)

$$\alpha^{\nu}_{lm} = \frac{a^{\nu}_{lm}}{\lambda_l + \lambda_m - \lambda_\nu} \quad (\nu,\ l,\ m = \mp 1,\ \mp 2). \tag{3.4}$$

We recall that $\lambda_{\mp 1}$ and $\lambda_{\mp 2}$ are found by means of (2.4), and a^{ν}_{lm} from (2.16) of the end of the preceding subsection. α^{ν}_{lm} are real.

3.4. General solution of the Cauchy problem. The general solution of system (3.2)

$$y_\nu = e^{\lambda_\nu \tau} y_\nu(0) \quad (\nu = \mp 1, \mp 2)$$

enables us to solve this problem for the initial system within the assumed approximation. By (3.3),

$$x_j(\tau) = e^{\lambda_j \tau} y_j(0) + \sum_{l,\, m = \mp 1,\, \mp 2} \alpha^j_{lm} y_l(0)\, y_m(0)\, e^{(\lambda_l + \lambda_m)\tau} + (3) \quad (j = \mp 1, \mp 2). \tag{4.1}$$

Substitution of the expressions obtained into formulas (2.11) yields the following general solution for the initial variables

$$\frac{1}{2} z_{-1} = \operatorname{Re} y_1(0) \cos \tau - \operatorname{Im} y_1(0) \sin \tau + \operatorname{Re} y_2(0) \cos \lambda\tau - \operatorname{Im} y_2(0) \sin \lambda\tau + \sum_{l,m} (\alpha^1_{lm} + \alpha^2_{lm}) \left[R_{lm} \cos \frac{1}{i} (\lambda_l + \lambda_m)\tau - I_{lm} \sin \frac{1}{i} (\lambda_l + \lambda_m)\tau \right],$$

$$\frac{1}{2} z_1 = -\omega_2 \operatorname{Re} y_1(0) \sin \tau - \omega_2 \operatorname{Im} y_1(0) \cos \tau - \omega_1 \operatorname{Re} y_2(0) \sin \lambda\tau - \omega_1 \operatorname{Im} y_2(0) \cos \lambda\tau - \sum_{l,m} (\omega_2 \alpha^1_{lm} + \omega_1 \alpha^2_{lm}) \left[R_{lm} \sin \frac{1}{i} (\lambda_l + \lambda_m)\tau + I_{lm} \cos \frac{1}{i} (\lambda_l + \lambda_m)\tau \right],$$

$$\frac{1}{2} z_{-2} = d_2 \operatorname{Re} y_1(0) \sin \tau + d_2 \operatorname{Im} y_1(0) \cos \tau + d_1 \operatorname{Re} y_2(0) \sin \lambda\tau + d_1 \operatorname{Im} y_2(0) \cos \lambda\tau + \sum_{l,m} (d_2 \alpha^1_{lm} + d_1 \alpha^2_{lm}) \left[R_{lm} \sin \frac{1}{i} (\lambda_l + \lambda_m)\tau + I_{lm} \cos \frac{1}{i} (\lambda_l + \lambda_m)\tau \right],$$

$$\frac{1}{2} z_2 = \omega_2 d_2 \operatorname{Re} y_1(0) \cos \tau - \omega_2 d_2 \operatorname{Im} y_1(0) \sin \tau + \omega_1 d_1 \operatorname{Re} y_2(0) \cos \lambda\tau - \omega_1 d_1 \operatorname{Im} y_2(0) \sin \lambda\tau + \sum_{l,m} (\omega_2 d_2 \alpha^1_{lm} + \omega_1 d_1 \alpha^2_{lm}) \left[R_{lm} \cos \frac{1}{i} (\lambda_l + \lambda_m)\tau - I_{lm} \sin \frac{1}{i} (\lambda_l + \lambda_m)\tau \right],$$

where for the sake of brevity

$$R_{lm} = \operatorname{Re} [y_l(0)\, y_m(0)] = \operatorname{Re} y_l(0) \operatorname{Re} y_m(0) - \operatorname{Im} y_l(0) \operatorname{Im} y_m(0),$$
$$I_{lm} = \operatorname{Im} [y_l(0)\, y_m(0)] = \operatorname{Re} y_l(0) \operatorname{Im} y_m(0) + \operatorname{Im} y_l(0) \operatorname{Re} y_m(0).$$

It remains to express the initial values $y_\nu(0)$ in terms of the initial values of the variables $z_j(0)$ (ν, $j = \mp 1, \mp 2$). Inverting the normalizing transformation (3.3), we arrive at

$$y_\nu(0) = x_\nu(0) - \sum_{j,h} \alpha^\nu_{jh} x_j(0) x_h(0) \quad (\nu = \mp 1, \mp 2),$$

and, substituting x_ν from (2.10), we obtain

$$\begin{aligned} y_\nu(0) = (-1)^\nu \left(\frac{d_2}{d_1\lambda}\right)^{|\nu|-1} & D z_{-1}(0) \\ & + \operatorname{sign}\nu\,(-1)^\nu\, i \left(\frac{d_2}{d_1}\right)^{|\nu|-1} D_1 z_1(0) \\ + \operatorname{sign}\nu\,(-1)^\nu\, i & \left(\frac{1}{\lambda}\right)^{|\nu|-1} D_2 z_{-2}(0) - (-1)^\nu D_3 z_2(0) \\ - \sum_{j,h} \alpha^\nu_{jh} \Bigg[(-1)^j \left(\frac{d_2}{d_1\lambda}\right)^{|j|-1} & D z_{-1}(0) + \operatorname{sign} j\,(-1)^j\, i \left(\frac{d_2}{d_1}\right)^{|j|-1} D_1 z_1(0) \\ + \operatorname{sign} j\,(-1)^j\, i & \left(\frac{1}{\lambda}\right)^{|j|-1} D_2 z_{-2}(0) - (-1)^j D_3 z_2(0) \Bigg] \\ \times \Bigg[(-1)^h \left(\frac{d_2}{d_1\lambda}\right)^{|h|-1} & D z_{-1}(0) + \operatorname{sign} h\,(-1)^h\, i \left(\frac{d_2}{d_1}\right)^{|h|-1} D_1 z_1(0) \\ + \operatorname{sign} h\,(-1)^h\, i & \left(\frac{1}{\lambda}\right)^{|h|-1} D_2 z_{-2}(0) - (-1)^h D_3 z_2(0) \Bigg] \\ & (\nu = \mp 1, \mp 2). \end{aligned}$$

Summation indices are ∓ 1, ∓ 2 everywhere and are independent of one another.

As an example, the formulas derived in [346b] were used to calculate trajectories in a model with parameters

$$k = 5.926 \text{ kgf/cm}, \quad m = 0.2635 \times 10^{-4} \text{ kgf}\cdot\text{cm}^{-1}\text{s}^2,$$
$$l = 2 \times 10^{-3} \text{ cm}, \quad p^2 = 22.51 \times 10^4 \text{ s}^{-1}, \quad \omega_0 = 800 \text{ s}^{-1}.$$

These values give $r_0 = 3.085 \times 10^{-3}$ cm.

The following initial conditions were chosen: the initial deviation from the equilibrium trajectory $z_{-1} = 0.3 r_0 = 0.9255 \times 10^{-3}$ cm; the initial angular velocity $\dot\varphi = 432$ s^{-1} was obtained from the equality of mass momenta for $r = r_0$ and $r_1 = 1.3 r_0$, whence $z_2 = 800 - 432 = 368$ s^{-1}. The initial values of the remaining variables were set to zero (z_1 and z_{-2}).

Substituting the numerical values of the parameters into (2.7) and (2.14), we obtain the coefficients a^ν_{jh} (ν, j, $h = \mp 1, \mp 2$) from (2.16) and α^ν_{jh} (ν, j, $h = \mp 1, \mp 2$) from (3.4). Then we calculate the

initial values $y_\nu(0)$ ($\nu = \mp 1, \mp 2$) and finally $z_{-1}(t)$ and $z_{-2}(t)$, which, if constant terms are neglected, become

$$z_{-1}(t) = 0.752 \times 10^{-3} \cos \omega_1 t + 0.175 \times 10^{-3} \cos \omega_2 t - [0.336 \cos 2\omega_2 t + 0.094 \cos 1.15\omega_2 t + 0.038 \cos 0.85\omega_2 t - 0.006 \cos 2\omega_1 t] \times 10^{-3} \text{ cm},$$

$$z_{-2}(t) = 0.257 \sin \omega_1 t + 0.082 \sin \omega_2 t - 2 \times 0.017 \sin 2\omega_2 t - 2 \times 0.002 \sin 1.15\omega_2 t + 2 \times 0.013 \sin 0.85\omega_2 t + 2 \times 0.001 \sin 2\omega_1 t \text{ cm}.$$

The above expressions demonstrate that the method of normal forms furnishes a more exact solution and enables us to derive trajectories of the centre of the shaft's cross section close to those realized in practice, while the linear approximation yields only elliptic trajectories rarely encountered in actual systems. The more exact solution involves high-frequency components; within the framework of the given model this indicates that stresses generated in the shaft material oscillate at frequencies exceeding the frequency of rotation. This may result in fatigue failure of a vertical flexible shaft operating in stable modes, which was mentioned in Subsection 3.1.

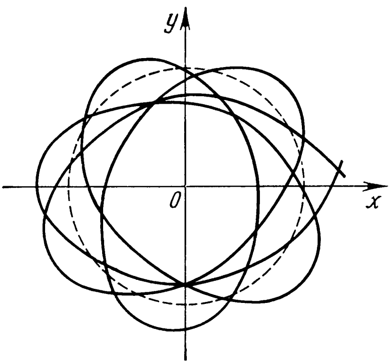

FIG. 16

The path traced by the point where the mass is elastically fixed on the shaft was calculated for the parameters of the model as given above. The time and, consequently, the period were found at the lowest frequency

$$t_j = j \frac{\pi}{48\omega_2} \quad (j = 0, 1, 2, \ldots, 96),$$

which corresponded to a rotation through $\omega_0 t_j = 15°$. For comparison, a circular trajectory of radius r_0 at "perturbation" $z_{-2} = 0$ is dashed in Fig. 16.

§ 4. Sixth-Order Systems

In this section we analyze resonances and normal forms of analytic autonomous (not necessarily conservative) sixth-order systems with three pairs of distinct pure imaginary eigenvalues of the matrix of the linear part. The section ends with an analysis of stability.

4.1. Solutions of the resonant equation. We consider the system defined above under the assumption that the linear part of the system is reduced to diagonal form (see Subsection 1.1)

$$\frac{dx_\nu}{d\tau}=\lambda_\nu x_\nu+\sum a^{\nu}_{jh}x_jx_h+\sum b^{\nu}_{jhk}x_jx_hx_k+\dots \tag{1.1}$$
$$(\nu=\mp 1,\quad \mp 2,\quad \mp 3).$$

Summation indices in this section always take on the values ∓ 1, ∓ 2, ∓ 3; $\lambda_{-\nu}=\overline{\lambda}_\nu$; without restricting the general character of the analysis, we assume that

$$\lambda_{-1}=-i,\quad \lambda_1=i,\quad \lambda_{-2}=-\mu i,\quad \lambda_2=\mu i,$$
$$\lambda_{-3}=-\lambda i,\quad \lambda_3=\lambda i\quad \left(i=\sqrt{-1},\quad 0<\lambda<\mu<1\right). \tag{1.2}$$

In general, the coefficients a^{ν}_{jh}, b^{ν}_{jhk}, ... are complex and symmetrized

$$a^{\nu}_{hj}=a^{\nu}_{jh},\quad b^{\nu}_{\{jhk\}}=\text{id.}\quad (\nu,\ j,\ h,\ k=\mp 1,\ \mp 2,\ \mp 3). \tag{1.3}$$

By the fundamental Brjuno theorem (see Chapter V, Subsection 1.2), there exists a reversible complex substitution of variables (a *normalizing transformation*)

$$x_j=y_j+\sum \alpha^{j}_{lm}y_ly_m+\sum \beta^{j}_{lmn}y_ly_my_n+\dots \tag{1.4}$$
$$(j=\mp 1,\ \mp 2,\ \mp 3)$$
$$(\alpha^{j}_{ml}=\alpha^{j}_{lm},\ \beta^{j}_{\{lmn\}}=\text{id.},\quad j,\ l,\ m,\ n=\mp 1,\ \mp 2,\ \mp 3)$$

that reduces system (1.1) to *normal form*

$$\frac{dy_\nu}{d\tau}=\lambda_\nu y_\nu+y_\nu\sum_{(\Lambda,\ Q)=0} g_{\nu Q}y_{-1}^{q_{-1}}y_1^{q_1}y_{-2}^{q_{-2}}y_2^{q_2}y_{-3}^{q_{-3}}y_3^{q_3} \tag{1.5}$$
$$(\nu=\mp 1,\ \mp 2,\ \mp 3),$$

where $q_{-1}, \dots, q_3$ are either integers or zeros; in addition, $q_\nu\geqslant -1$, while the remaining q_j are nonnegative, $\sum q_h\geqslant 1$. A normal form includes only those resonant terms whose exponents satisfy the *resonant equation* $(\mathbf{\Lambda},\ \mathbf{Q})=0$; in detailed form,

$$q_1-q_{-1}+\mu\,(q_2-q_{-2})+\lambda\,(q_3-q_{-3})=0. \tag{1.6}$$

We consider the possibility of having in (1.5) rth-degree resonant terms for which

$$q_{-1}+q_1+q_{-2}+q_2+q_{-3}+q_3=r-1\quad (r\geqslant 2). \tag{1.7}$$

For any λ and μ from (1.2) and any odd $r\geqslant 3$, the resonant equation (1.6) has the *trivial solution*

$$q_{-1}=q_1,\quad q_{-2}=q_2,\quad q_{-3}=q_3. \tag{1.8}$$

Now we consider *three semitrivial solutions*, when one of the expressions in (1.6) in parentheses vanishes

$$q_{-1}=q_1, \quad \frac{\lambda}{\mu}=\frac{q_{-2}-q_2}{q_3-q_{-3}}; \tag{1.9}$$

$$q_{-2}=q_2, \quad \lambda=\frac{q_{-1}-q_1}{q_3-q_{-3}}; \tag{1.10}$$

$$q_{-3}=q_3, \quad \mu=\frac{q_{-1}-q_1}{q_2-q_{-2}}. \tag{1.11}$$

When two expressions in (1.6) in parentheses vanish, the third vanishes as well, which corresponds to the trivial solution. Thus it remains to find the *nontrivial solution*, when all three are distinct from zero

$$\frac{q_{-3}-q_3}{q_1-q_{-1}}\lambda+\frac{q_{-2}-q_2}{q_1-q_{-1}}\mu=1. \tag{1.12}$$

The trivial solution is impossible *for quadratic terms* ($r = 2$). Semitrivial solutions are possible only for specific values of λ and μ and they give for the resonant terms of equations (1.5), respectively,

$$\frac{\lambda}{\mu}=\frac{1}{2}: \quad \mathbf{Q}_{-1}=\mathbf{Q}_1=\mathbf{0}, \quad \mathbf{Q}_{-2}=(0,\ 0,\ -1,\ 0,\ 2,\ 0),$$
$$\mathbf{Q}_2=(0,\ 0,\ 0,\ -1,\ 0,\ 2), \quad \mathbf{Q}_{-3}=(0,\ 0,\ 1,\ 0,\ -1,\ 1),$$
$$\mathbf{Q}_3=(0,\ 0,\ 0,\ 1,\ 1,\ -1);$$

$$\lambda=\frac{1}{2}: \quad \mathbf{Q}_{-1}=(-1,\ 0,\ 0,\ 0,\ 2,\ 0), \quad \mathbf{Q}_1=(0,\ -1,\ 0,\ 0,\ 0,\ 2),$$
$$\mathbf{Q}_{-2}=\mathbf{Q}_2=\mathbf{0}, \quad \mathbf{Q}_{-3}=(1,\ 0,\ 0,\ 0,\ -1,\ 1),$$
$$\mathbf{Q}_3=(0,\ 1,\ 0,\ 0,\ 1,\ -1);$$

$$\mu=\frac{1}{2}: \quad \mathbf{Q}_{-1}=(-1,\ 0,\ 2,\ 0,\ 0,\ 0), \quad \mathbf{Q}_1=(0,\ -1,\ 0,\ 2,\ 0,\ 0),$$
$$\mathbf{Q}_{-2}=(1,\ 0,\ -1,\ 1,\ 0,\ 0), \quad \mathbf{Q}_2=(0,\ 1,\ 1,\ -1,\ 0,\ 0),$$
$$\mathbf{Q}_{-3}=\mathbf{Q}_3=\mathbf{0}.$$

For the nontrivial solution we obtain

$$\lambda+\mu=1: \quad \mathbf{Q}_{-1}=(-1, 0, 1, 0, 1, 0), \quad \mathbf{Q}_1=(0, -1, 0, 1, 0, 1),$$
$$\mathbf{Q}_{-2}=(1, 0, -1, 0, 0, 1), \quad \mathbf{Q}_2=(0, 1, 0, -1, 1, 0),$$
$$\mathbf{Q}_{-3}=(1, 0, 0, 1, -1, 0), \quad \mathbf{Q}_3=(0, 1, 1, 0, 0, -1).$$

In the case of *third-power terms* ($r = 3$), the trivial solution (1.8) yields

$$\mathbf{Q}_\nu = (1, 1, 0, 0, 0, 0), \quad \mathbf{Q}_\nu = (0, 0, 1, 1, 0, 0),$$
$$\mathbf{Q}_\nu = (0, 0, 0, 0, 1, 1) \qquad (\nu = \mp 1, \mp 2, \mp 3)$$

for any λ and μ from (1.2).

All the remaining solutions are possible only for specific values of λ and μ from (1.2). They are given below, together with the exponents $\mathbf{Q}_\nu$ found in accordance with equations (1.5). The semitrivial solutions

$$\frac{\lambda}{\mu} = \frac{1}{3}: \quad \mathbf{Q}_{-1} = \mathbf{Q}_1 = \mathbf{0}, \qquad \mathbf{Q}_{-2} = (0, 0, -1, 0, 3, 0),$$
$$\mathbf{Q}_2 = (0, 0, 0, -1, 0, 3), \quad \mathbf{Q}_{-3} = (0, 0, 1, 0, -1, 2),$$
$$\mathbf{Q}_3 = (0, 0, 0, 1, 2, -1);$$
$$\lambda = \frac{1}{3}: \quad \mathbf{Q}_{-1} = (-1, 0, 0, 0, 3, 0), \quad \mathbf{Q}_1 = (0, -1, 0, 0, 0, 3),$$
$$\mathbf{Q}_{-2} = \mathbf{Q}_2 = \mathbf{0}, \qquad \mathbf{Q}_{-3} = (1, 0, 0, 0, -1, 2),$$
$$\mathbf{Q}_3 = (0, 1, 0, 0, 2, -1);$$
$$\mu = \frac{1}{3}: \quad \mathbf{Q}_{-1} = (-1, 0, 3, 0, 0, 0), \quad \mathbf{Q}_1 = (0, -1, 0, 3, 0, 0),$$
$$\mathbf{Q}_{-2} = (1, 0, -1, 2, 0, 0), \quad \mathbf{Q}_2 = (0, 1, 2, -1, 0, 0),$$
$$\mathbf{Q}_{-3} = \mathbf{Q}_3 = \mathbf{0}.$$

And, finally. the nontrivial solutions yield

$$2\lambda + \mu = 1: \quad \mathbf{Q}_{-1} = (-1, 0, 1, 0, 2, 0), \quad \mathbf{Q}_1 = (0, -1, 0, 1, 0, 2),$$
$$\mathbf{Q}_{-2} = (1, 0, -1, 0, 0, 2), \quad \mathbf{Q}_2 = (0, 1, 0, -1, 2, 0),$$
$$\mathbf{Q}_{-3} = (1, 0, 0, 1, -1, 1), \quad \mathbf{Q}_3 = (0, 1, 1, 0, 1, -1);$$
$$\lambda + 2\mu = 1: \quad \mathbf{Q}_{-1} = (-1, 0, 2, 0, 1, 0), \quad \mathbf{Q}_1 = (0, -1, 0, 2, 0, 1),$$
$$\mathbf{Q}_{-2} = (1, 0, -1, 1, 0, 1), \quad \mathbf{Q}_2 = (0, 1, 1, -1, 1, 0),$$
$$\mathbf{Q}_{-3} = (1, 0, 0, 2, -1, 0), \quad \mathbf{Q}_3 = (0, 1, 2, 0, 0, -1);$$
$$2\mu - \lambda = 1: \quad \mathbf{Q}_{-1} = (-1, 0, 2, 0, 0, 1), \quad \mathbf{Q}_1 = (0, -1, 0, 2, 1, 0),$$
$$\mathbf{Q}_{-2} = (1, 0, -1, 1, 1, 0), \quad \mathbf{Q}_2 = (0, 1, 1, -1, 0, 1),$$
$$\mathbf{Q}_{-3} = (0, 1, 2, 0, -1, 0), \quad \mathbf{Q}_3 = (1, 0, 0, 2, 0, -1).$$

The resonances are given below in Fig. 17 for $r = 2$ and in Fig. 18 for $r = 3$.

4.2. Normal forms. If equations (1.5) are truncated to terms of power not higher than three, the fundamental Brjuno theorem (V, 1.2) yields the following normal forms:

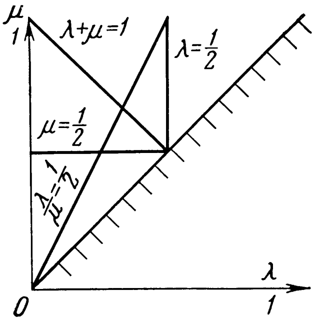

FIG. 17

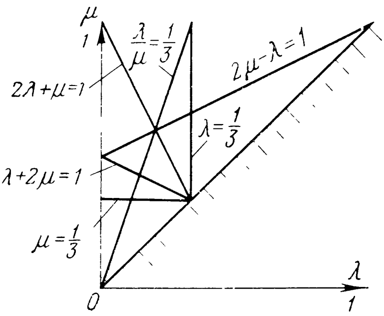

FIG. 18

(a) With no resonances, that is, for

$$\lambda \neq \frac{1}{2},\ \frac{1}{3};\quad \mu \neq \frac{1}{2},\ \frac{1}{3};\quad \frac{\lambda}{\mu} \neq \frac{1}{2},\ \frac{1}{3};$$
$$\lambda+\mu\neq 1,\quad 2\lambda+\mu\neq 1,\quad 2\mu\pm\lambda\neq 1$$

(the *general case*, when λ and μ from (1.2) do not lie on the straight lines of Figs. 17 and 18),

$$\frac{dy_\nu}{d\tau}=\lambda_\nu y_\nu+g_1^\nu y_\nu y_{-1}y_1+g_2^\nu y_\nu y_{-2}y_2+g_3^\nu y_\nu y_{-3}y_3 \tag{2.1}$$
$$(\nu=\mp 1,\ \mp 2,\ \mp 3).$$

(b) For the resonances appearing in semitrivial solutions (1.9)-(1.11) of equation (1.6), one term has to be added to each equation of (2.1), respectively,

$$\frac{\lambda}{\mu}=\frac{1}{2}:\quad 0,\ 0,\ f_{-2}y_{-3}^2,\ f_2y_3^2,\ f_{-3}y_{-2}y_3,\ f_3y_2y_{-3};$$

$$\lambda=\frac{1}{2}:\quad e_{-1}y_{-3}^2,\ e_1y_3^2,\ 0,\ 0,\ e_{-3}y_{-1}y_3,\ e_3y_1y_{-3};$$

$$\mu=\frac{1}{2}:\quad d_{-1}y_{-2}^2,\ d_1y_2^2,\ d_{-2}y_{-1}y_2,\ d_2y_1y_{-2},\ 0,\ 0;$$

$$\frac{\lambda}{\mu}=\frac{1}{3}:\ 0,\ 0,\ c_{-2}y_{-3}^3,\ c_2y_3^3,\ c_{-3}y_{-2}y_3^2,\ c_3y_2y_{-3}^2;$$

$$\lambda=\frac{1}{3}:\quad b_{-1}y_{-3}^3,\ b_1y_3^3,\ 0,\ 0,\ b_{-3}y_{-1}y_3^2,\ b_3y_1y_{-3}^2;$$

$$\mu=\frac{1}{3}:\quad a_{-1}y_{-2}^3,\ a_1y_2^3,\ a_{-2}y_{-1}y_2^2,\ a_2y_1y_{-2}^2,\ 0,\ 0.$$

(c) For the resonances appearing in nontrivial solutions (1.12) of equation (1.6), each equation of (2.1) must be supplemented by one term, respectively,

$$\lambda+\mu=1: \quad h_{-1}y_{-2}y_{-3},\ h_1y_2y_3,\ h_{-2}y_{-1}y_3,\ h_2y_1y_{-3},\ h_{-3}y_{-1}y_2,\ h_3y_1y_{-2};$$

$$2\lambda+\mu=1: \quad i_{-1}y_{-2}y_{-3}^2,\ i_1y_2y_3^2,\ i_{-2}y_{-1}y_3^2,\ i_2y_1y_{-3}^2,\ i_{-3}y_{-1}y_2y_3,\ i_3y_1y_{-2}y_{-3};$$

$$\lambda+2\mu=1: \quad j_{-1}y_{-2}^2y_{-3},\ j_1y_2^2y_3,\ j_{-2}y_{-1}y_2y_3,\ j_2y_1y_{-2}y_{-3},\ j_{-3}y_{-1}y_2^2,\ j_3y_1y_{-2}^2;$$

$$2\mu-\lambda=1: \quad k_{-1}y_{-2}^2y_3,\ k_1y_2^2y_{-3},\ k_{-2}y_{-1}y_2y_{-3},\ k_2y_1y_{-2}y_3,\ k_{-3}y_1y_{-2}^2,\ k_3y_{-1}y_2^2.$$

Remark. For the values of λ and μ belonging to two or more resonances (this corresponds to intersections of the sets of straight lines in Figs. 17 and 18), superposition of two or more additional terms occurs in equations (2.1).

In the real case, $y_{-\nu}=\overline{y}_\nu$ ($\nu=\mp 1, \mp 2, \mp 3$), it would be sufficient to write the normal forms only for $\nu = 1, 2, 3$ (or only for $\nu = -1, -2, -3$). Here, however, we also cover the case in which the variables in the initial system (1.1) are not complex conjugate.

4.3. Calculation of coefficients of normalizing transformation and normal forms. In the general case (a) of Subsection 4.2, the normalizing transformation (1.4) reduces system (1.1) to the normal form (2.1), and additional terms appear in (2.1) in the resonant cases (b) and (c). Reducing the normal form to symmetrized representation (V, 3, 1.3a), we arrive at the fundamental identities (V, 3, 1.6), and then follow the alternative of Chapter V, Subsection 3.2.

Let us eliminate the resonances appearing in the quadratic terms (see Subsection 4.1), that is, assume that

$$\frac{\lambda}{\mu}\neq\frac{1}{2}; \quad \lambda,\ \mu\neq\frac{1}{2}; \quad \lambda+\mu\neq 1 \tag{3.1}$$

(see also (1.2)). Then $\lambda_\nu \neq \lambda_l + \lambda_m$ (ν, l, $m = \mp 1, \mp 2, \mp 3$) and formula (V, 3, 2.2) is valid for the quadratic coefficients of the normalizing transformation

$$\alpha_{lm}^{\nu}=\frac{a_{lm}^{\nu}}{\lambda_l+\lambda_m-\lambda_\nu} \quad (\nu,\ l,\ m=\mp 1,\ \mp 2,\ \mp 3) \tag{3.2}$$

(with the restrictions imposed in (3.1)).

When $\dfrac{\lambda}{\mu}=\dfrac{1}{2}$ (see case (b) of Subsection 4.2), then $\lambda_\nu=\lambda_l+\lambda_m$ if and only if $\nu, l, m = -2, -3, -3;\ 2, 3, 3;\ -3, \{-2, 3\};\ 3, \{2, -3\}$. We choose

$$\alpha_{-3-3}^{-2},\quad \alpha_{33}^{2},\quad \alpha_{\{-23\}}^{-3},\quad \alpha_{\{2-3\}}^{3}$$

arbitrarily (it is preferable to determine them from (3.2) on the basis of continuity when $\dfrac{\lambda}{\mu}\to\dfrac{1}{2}$, if this is possible, or set them to

zero otherwise). The remaining α^{ν}_{lm} are given by (3.2). The coefficients of the quadratic terms appearing when $\frac{\lambda}{\mu} = \frac{1}{2}$ (see case (b) of Subsection 4.2) are given by formulas (V, 3, 2.4)

$$f_{-2} = \varphi^{-2}_{-3-3} = a^{-2}_{-3-3}, \quad f_2 = \varphi^{2}_{33} = a^{2}_{33},$$
$$f_{-3} = 2\varphi^{-3}_{-23} = 2a^{-3}_{-23}, \quad f_3 = 2\varphi^{3}_{2-3} = 2a^{3}_{2-3}.$$

Similarly, when $\lambda = \frac{1}{2}$, we choose

$$\alpha^{-1}_{-3-3}, \quad \alpha^{1}_{33}, \quad \alpha^{-3}_{-13}, \quad \alpha^{3}_{1-3}$$

arbitrarily and determine

$$e_{-1} = a^{-1}_{-3-3}, \quad e_1 = a^{1}_{33}, \quad e_{-3} = 2a^{-3}_{-13}, \quad e_3 = 2a^{3}_{1-3};$$

when $\mu = \frac{1}{2}$, we choose

$$\alpha^{-1}_{-2-2}, \quad \alpha^{1}_{22}, \quad \alpha^{-2}_{\{-12\}}, \quad \alpha^{2}_{\{1-2\}}$$

arbitrarily and determine

$$d_{-1} = a^{-1}_{-2-2}, \quad d_1 = a^{1}_{22}, \quad d_{-2} = 2a^{-2}_{-12}, \quad d_2 = 2a^{2}_{1-2};$$

and when $\lambda + \mu = 1$, we choose

$$\alpha^{-1}_{\{-2-3\}}, \quad \alpha^{1}_{\{23\}}, \quad \alpha^{-2}_{\{-13\}}, \quad \alpha^{2}_{\{1-3\}}, \quad \alpha^{-3}_{\{-12\}}, \quad \alpha^{3}_{\{1-2\}}$$

arbitrarily and determine

$$h_{-1} = 2a^{-1}_{-2-3}, \quad h_1 = 2a^{1}_{23}, \quad h_{-2} = 2a^{-2}_{-13},$$
$$h_2 = 2a^{2}_{1-3}, \quad h_{-3} = 2a^{-3}_{-12}, \quad h_3 = 2a^{3}_{1-2}.$$

In order to determine the cubic terms, we must first eliminate the resonances that appear, that is, assume that

$$\frac{\lambda}{\mu} \neq \frac{1}{3}; \quad \lambda, \mu \neq \frac{1}{3}; \quad 2\lambda + \mu \neq 1; \quad 2\mu \pm \lambda \neq 1. \tag{3.3}$$

The values of ν, l, m, and p for which $\lambda_\nu = \lambda_l + \lambda_m + \lambda_p$ are

$$\nu, l, m, p = \nu, \{l, -l, \nu\} \quad (\nu, l = \mp 1, \mp 2, \mp 3). \tag{3.4}$$

We also set

$$\beta^{\nu}_{\{l-l\nu\}} = 0 \quad (\nu, l = \mp 1, \mp 2, \mp 3). \tag{3.5}$$

Formula (V, 3, 2.3) holds for the remaining values of ν, l, m, and p, that is,

$$\beta^{\nu}_{lmp} = \frac{1}{\lambda_l + \lambda_m + \lambda_p - \lambda_\nu}\Big[b^{\nu}_{lmp} + \frac{2}{3} \sum_{j=\mp1,\,\mp2,\,\mp3} (a^{\nu}_{lj}\alpha^{j}_{mp} + a^{\nu}_{mj}\alpha^{j}_{pl} + a^{\nu}_{jp}\alpha^{j}_{lm}) \Big] \tag{3.6}$$

$$(\nu,\ l,\ m,\ p = \mp1,\ \mp2,\ \mp3;\quad l,\ m,\ p \neq \{l,\ -l,\ \nu\})$$

with the restrictions given in (3.3). Formulas (V, 3, 2.6) under conditions (3.1) yield the coefficients of the cubic terms in (2.1) corresponding to the subscripts selected in (3.4)

$$g^{\nu}_{|\nu|} = 3\chi^{\nu}_{\nu\nu-\nu} = 3b^{\nu}_{\nu\nu-\nu} + 2 \sum_{j=\mp1,\,\mp2,\,\mp3} (2a^{\nu}_{\nu j}\alpha^{j}_{\nu-\nu} + a^{\nu}_{-\nu j}\alpha^{j}_{\nu\nu})$$

$$(\nu = \mp1,\ \mp2,\ \mp3),$$

$$g^{\nu}_{h} = 6\chi^{\nu}_{\nu h-h} = 6b^{\nu}_{\nu h-h} + 4 \sum_{j} (a^{\nu}_{\nu j}\alpha^{j}_{h-h} + a^{\nu}_{hj}\alpha^{j}_{-h\nu} + a^{\nu}_{-hj}\alpha^{j}_{\nu h}) \tag{3.7}$$

$$(\nu = \mp1,\ \mp2,\ \mp3;\quad h = 1,\ 2,\ 3;\quad h \neq |\nu|).$$

When $\frac{\lambda}{\mu} = \frac{1}{2}$, λ, $\mu = \frac{1}{2}$, $\lambda + \mu = 1$, formulas (V, 3, 2.5) must be applied.

It remains to analyze the cases omitted in (3.3). When $\frac{\lambda}{\mu} = \frac{1}{3}$, then the equality $\lambda_\nu = \lambda_l + \lambda_m + \lambda_p$ holds, in addition to the subscripts given in (3.4), for

$$\nu,\ l,\ m,\ p = -2,\ -3,\ -3,\ -3;\quad 2,\ 3,\ 3,\ 3;$$
$$-3,\ \{-2,\ 3,\ 3\};\quad 3,\ \{2,\ -3,\ -3\},$$

and we choose

$$\beta^{-2}_{-3-3-3},\quad \beta^{2}_{333},\quad \beta^{-3}_{\{-233\}},\quad \beta^{3}_{\{2-3-3\}}$$

arbitrarily (it is preferable to determine them on the basis of continuity by means of (3.6) when $\frac{\lambda}{\mu} \to \frac{1}{3}$, if this is possible, or set them to zero otherwise). The coefficients of the cubic terms appearing when $\frac{\lambda}{\mu} = \frac{1}{3}$ (see case (b) of Subsection 4.2) will be determined in general from formulas (V, 3, 2.5) (or from (V, 3, 2.6) when $\mu = \frac{1}{2}, \frac{3}{4}$, since (3.1) holds and the normal form contains no quadratic terms). Namely,

$$c_{-2} = \chi^{-2}_{-3-3-3},\quad c_2 = \chi^{2}_{333},\quad c_{-3} = 3\chi^{-3}_{-233},\quad c_3 = 3\chi^{3}_{2-3-3}.$$

Similarly, when $\lambda = \frac{1}{3}$, we choose

$$\beta^{-1}_{-3-3-3}, \quad \beta^{1}_{333}, \quad \beta^{-3}_{\{-133\}}, \quad \beta^{3}_{\{1-3-3\}}$$

arbitrarily and determine

$$b_{-1} = \chi^{-1}_{-3-3-3}, \quad b_1 = \chi^{1}_{333}, \quad b_{-3} = 3\chi^{-3}_{-133}, \quad b_3 = 3\chi^{3}_{1-3-3}$$

from formulas (V, 3, 2.6) when $\mu \neq \frac{1}{3}, \frac{2}{3}$ or from formulas (V, 3, 2.5) when $\mu = \frac{1}{2}, \frac{2}{3}$.

When $\mu = \frac{1}{3}$, we choose

$$\beta^{-1}_{-2-2-2}, \quad \beta^{1}_{222}, \quad \beta^{-2}_{-122}, \quad \beta^{2}_{1-2-2}$$

arbitrarily and determine

$$a_{-1} = \chi^{-1}_{-2-2-2}, \quad a_1 = \chi^{1}_{222}, \quad a_{-2} = 3\chi^{-2}_{-122}, \quad a_2 = 3\chi^{2}_{1-2-2}$$

from formulas (V, 3, 2.6) when $\lambda \neq \frac{1}{6}$ or from formulas (V, 3, 2.5) when $\lambda = \frac{1}{6}$.

When $2\lambda + \mu = 1$, we choose

$$\beta^{-1}_{\{-2-3-3\}}, \quad \beta^{1}_{\{233\}}, \quad \beta^{-2}_{\{-133\}}, \quad \beta^{2}_{\{1-3-3\}}, \quad \beta^{-3}_{\{-123\}}, \quad \beta^{3}_{\{1-2-3\}}$$

arbitrarily and determine

$$i_{-1} = 3\chi^{-1}_{-2-3-3}, \quad i_1 = 3\chi^{1}_{233}, \quad i_{-2} = 3\chi^{-2}_{-133},$$
$$i_2 = 3\chi^{2}_{1-3-3}, \quad i_{-3} = 6\chi^{-3}_{-123}, \quad i_3 = 6\chi^{3}_{1-2-3}$$

from formulas (V, 3, 2.6) when $\lambda \neq \frac{1}{4}$ or from formulas (V, 3, 2.5) when $\lambda = \frac{1}{4}$.

When $\lambda + 2\mu = 1$, we choose

$$\beta^{-1}_{\{-2-2-3\}}, \quad \beta^{1}_{\{223\}}, \quad \beta^{-2}_{\{-123\}}, \quad \beta^{2}_{\{1-2-3\}}, \quad \beta^{-3}_{\{-122\}}, \quad \beta^{3}_{\{1-2-2\}}$$

arbitrarily and determine

$$j_{-1} = 3\chi^{-1}_{-2-2-3}, \quad j_1 = 3\chi^{1}_{223}, \quad j_{-2} = 6\chi^{-2}_{-123},$$
$$j_2 = 6\chi^{2}_{1-2-3}, \quad j_{-3} = 3\chi^{-3}_{-122}, \quad j_3 = 3\chi^{3}_{1-2-2}$$

from formulas (V, 3, 2.6) when $\lambda \neq \frac{1}{4}$ or from formulas (V, 3, 2.5) when $\lambda = \frac{1}{4}$.

Finally, when $2\mu-\lambda=1$, we choose

$$\beta^{-1}_{\{-2-23\}},\quad \beta^{1}_{\{22-3\}},\quad \beta^{-2}_{\{-12-3\}},\quad \beta^{2}_{\{1-23\}},\quad \beta^{-3}_{\{1-2-2\}},\quad \beta^{3}_{\{-122\}}$$

arbitrarily and determine

$$k_{-1}=3\chi^{-1}_{-2-23},\quad k_1=3\chi^{1}_{22-3},\quad k_{-2}=6\chi^{-2}_{-12-3},$$
$$k_2=6\chi^{2}_{1-23},\quad k_{-3}=3\chi^{-3}_{1-2-2},\quad k_3=3\chi^{3}_{-122}$$

from formulas (V, 3, 2.6) when $\lambda\neq\frac{1}{3}, \frac{1}{2}$ or from formulas (V, 3, 2.5) when $\lambda=\frac{1}{3}, \frac{1}{2}$.

When β^{ν}_{lmp} have to be chosen, it is preferable to determine them on the basis of continuity from (3.6) (when possible) or set them to zero.

4.4. Stability in the third approximation. The Molchanov criterion. We consider the general case (a) of Subsection 4.2, assuming the initial system (1.1) to be real. This means that in (2.1) not only $\lambda_{-k}=\bar{\lambda}_k$, but also that

$$y_{-k}=\overline{y_k},\quad g_r^{-k}=\overline{g_r^k}\quad (k,\ r=1,\ 2,\ 3).$$

Multiplying equations (2.1) by $y_{-\nu}$ and adding them pairwise, we arrive at a system of real equations

$$\frac{d\eta_k}{d\tau}=-\eta_k\sum_{\alpha=1}^{3}E_{k\alpha}\eta_\alpha\quad (k=1,\ 2,\ 3), \tag{4.1}$$

where

$$\eta_k=|y_k|^2\geqslant 0,\quad E_{k\alpha}=-2\operatorname{Re} g_\alpha^k\quad (k,\ \alpha=1,\ 2,\ 3). \tag{4.2}$$

System (4.1) was analyzed by Molchanov [329b] for arbitrary $k\geqslant 2$. The case $k=2$ was presented in Subsection 1.4. Let us take up the case $k=3$, following [329b].

If all variables η_α in (4.1), except one, are set to zero, we obtain the necessary conditions for stability of the trivial solution of the real system (1.1) in the general case

$$E_{kk}>0\quad (k=1,\ 2,\ 3).$$

A sufficient condition for stability is the positive definiteness of the matrix

$$\left\|\frac{1}{2}(E_{k\alpha}+E_{\alpha k})\right\|_1^3.$$

This is readily seen if all the equations of (4.1) are added.

Now we wish to consider the necessary and sufficient conditions for stability of the trivial solution of system (4.1). Since the variables η_1, η_2, and η_3 are nonnegative, we must analyze stability only within a cone $\eta_k \geqslant 0$ ($k = 1, 2, 3$). Solutions of the type $\eta_k = \eta_k^0 \eta(t)$ are said to be the *invariant rays* of system (4.1). Substitution into (4.1) yields

$$\frac{d\eta}{d\tau} = -E\eta^2, \quad \eta(0) = 1, \tag{4.3}$$

$$\eta_k^0 \left[\sum_{\alpha=1}^{3} E_{k\alpha}\eta_\alpha^0 - E \right] = 0 \quad (k = 1,\ 2,\ 3). \tag{4.4}$$

Here E is a parameter similar to the eigenvalue in the linear systems; by (4.4) stability is shown to be determined by the sign of E.

In order to find the invariant rays of system (4.1), we retain only the second multiplier in each equation of (4.4); this yields the fundamental system of linear equations

$$\sum_{\alpha=1}^{3} E_{k\alpha}\eta_\alpha^0 = E \quad (k = 1,\ 2,\ 3). \tag{4.5}$$

If the matrix

$$\| E_{k\alpha} \|_1^3 \tag{4.6}$$

is nonsingular, then system (4.5) has a unique solution for any E. These solutions fill an invariant line formed by one stable ($E > 0$) and one unstable ($E < 0$) rays. If matrix (4.6) is singular, then a solution exists, within the proportionality factor, only for $E = 0$ (the neutral invariant line). All solutions of the nonlinear algebraic system (4.4) can be obtained by retaining in each of the equations either the first or the second multiplier. The total number of solutions is eight, including the identity solution $\eta_1 = \eta_2 = \eta_3 = 0$ analyzed above. This procedure corresponds to an independent investigation of system (4.1) on each of the three faces of the cone $\eta_k \geqslant 0$ ($k = 1,\ 2,\ 3$).

The Molchanov criterion [329b]. *For the trivial solution of system* (4.1) *to be stable, it is necessary and sufficient that no neutral or unstable ray be located within or on the faces of the cone* $\eta_k \geqslant 0$ ($k = 1, 2, 3$).

Necessity was proved above. A proof of sufficiency is outlined in [329b].

Remark. In a number of problems (see Subsections 1.1 and 2.6) all g_r^k $(k, r = 1, 2, 3)$ are pure imaginary. Then, by equations (4.1),

$$|y_k|^2 = c_k \quad (k = 1, 2, 3)$$

are the first integrals of system (2.1) and can be used to construct the Lyapunov function by the Chetaev method ([40a], Ch. II, Sec. 10) of linear combination of integrals. The Lyapunov function constructed in this manner, however, is determined within third-power terms and can only furnish conclusions on the formal stability [238e] of the trivial solution of system (1.1).

CHAPTER IX

OSCILLATIONS OF A HEAVY SOLID BODY WITH A FIXED POINT ABOUT THE LOWER EQUILIBRIUM POSITION

§ 1. Case of Centroid Located in a Principal Plane of the Ellipsoid of Inertia with Respect to a Fixed Point

The differential equations of oscillations of a heavy solid body with one fixed point about the lower equilibrium position are transformed in Subsections 1.1, 1.2, and 1.5 for the case specified in the heading of this section. It is preferable to apply known methods to the transformed equations; this is demonstrated by an example in which the method of successive approximations is used.

1.1. Reduction to diagonal form. We consider a nonsymmetric heavy solid body in which the centroid (centre of gravity) G lies in one of the principal planes of inertia for the fixed point O. Without restricting the general character of the analysis, we orient the principal axes $Oxyz$ of the ellipsoid of inertia in such a manner that $x_G \geqslant 0$, $y_G = 0$, and $z_G \geqslant 0$ ($OG^2 = x_G^2 + z_G^2 > 0$). Euler's equations are

$$\frac{dp}{dt} = \frac{1-c}{a} qr - \frac{Mgl}{aB} \zeta\gamma', \quad \frac{dq}{dt} = (c-a) rp + \frac{Mgl}{B} (\zeta\gamma - \xi\gamma''),$$

$$\frac{dr}{dt} = \frac{a-1}{c} pq + \frac{Mgl}{cB} \xi\gamma' \quad \left(a = \frac{A}{B}, \quad c = \frac{C}{B}, \quad \xi = \frac{x_G}{l}, \quad \zeta = \frac{z_G}{l}\right),$$

where $l = OG$ and the fixed axis Oz^* is directed downward, and γ, γ', and γ'' are its direction cosines. We wish to analyze oscillations about the lower equilibrium position ($\gamma = \xi$, $\gamma' = 0$, and $\gamma'' = \zeta$) and therefore assume that at every moment during the motion

$$\gamma = \xi + \Gamma, \quad \gamma' = \Gamma', \quad \gamma'' = \zeta + \Gamma''.$$

Let us introduce dimensionless variables and dimensionless time

$$P = \frac{p}{\tilde{\nu}}, \quad Q = \frac{q}{\tilde{\nu}}, \quad R = \frac{r}{\tilde{\nu}}, \quad \tau = \tilde{\nu} t \quad \left(\tilde{\nu} = \sqrt{\frac{Mgl}{B}}\right)$$

(note that $\tilde{\nu}$ stands for the frequency of pendulum oscillations about the axis Oy if it is horizontal). The Euler-Poisson equations can be

written as

$$\frac{dP}{d\tau}=-\frac{\zeta}{a}\Gamma'+\frac{1-c}{a}QR,\qquad \frac{d\Gamma}{d\tau}=-\zeta Q+R\Gamma'-Q\Gamma'',$$
$$\frac{dQ}{d\tau}=\zeta\Gamma-\xi\Gamma''+(c-a)\,RP,\quad \frac{d\Gamma'}{d\tau}=\zeta P-\xi R+P\Gamma''-R\Gamma,$$
$$\frac{dR}{d\tau}=\frac{\xi}{c}\Gamma'+\frac{a-1}{c}PQ,\qquad \frac{d\Gamma''}{d\tau}=\xi Q+Q\Gamma-P\Gamma'. \tag{1.1}$$

The eigenvalues of the matrix of the linear part of system (1.1) are 0, 0, $-i$, $+i$, $-\lambda i$, $+\lambda i$, and

$$\lambda=+\sqrt{\frac{1}{c}\xi^2+\frac{1}{a}\zeta^2}\qquad(\xi^2+\zeta^2=1), \tag{1.1a}$$

where the zero eigenvalue corresponds to simple elementary divisors. The matrix $\mathbf{S}$ having for its elements the correspondingly arranged eigenvectors of the matrix of the linear part of system (1.1) and its inverse matrix $\mathbf{S}^{-1}$ are

$$\mathbf{S}=\begin{Vmatrix} 0 & \xi & 0 & 0 & -i\frac{\zeta}{a\lambda} & i\frac{\zeta}{a\lambda} \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & \zeta & 0 & 0 & i\frac{\xi}{c\lambda} & -i\frac{\xi}{c\lambda} \\ \xi & 0 & -i\zeta & i\zeta & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ \zeta & 0 & i\xi & -i\xi & 0 & 0 \end{Vmatrix},$$

$$\mathbf{S}^{-1}=\begin{Vmatrix} 0 & 0 & 0 & \xi & 0 & 0 \\ \frac{\xi}{c\lambda^2} & 0 & \frac{\zeta}{a\lambda^2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & i\frac{\zeta}{2} & 0 & -i\frac{\xi}{2} \\ 0 & \frac{1}{2} & 0 & -i\frac{\zeta}{2} & 0 & i\frac{\xi}{2} \\ i\frac{\zeta}{2\lambda} & 0 & -i\frac{\xi}{2\lambda} & 0 & \frac{1}{2} & 0 \\ -i\frac{\zeta}{2\lambda} & 0 & i\frac{\xi}{2\lambda} & 0 & \frac{1}{2} & 0 \end{Vmatrix}. \tag{1.2}$$

We denote by $\mathbf{u}$ a vector with the components P, Q, R, Γ, Γ', and Γ'', and by $\mathbf{x}$ a vector with the components $x_1, \ldots, x_6$. System

(1.1) can be presented in the form

$$\frac{d\mathbf{u}}{d\tau} = \mathbf{A}\mathbf{u} + \mathbf{g}(\mathbf{u}),$$

where $\mathbf{g}(\mathbf{u})$ is a vector-function composed of the nonlinear terms of system (1.1). The substitution $\mathbf{u} = \mathbf{S}\mathbf{x}$ reduces system (1.1) to diagonal form

$$\frac{d\mathbf{x}}{d\tau} = \mathbf{S}^{-1}\mathbf{A}\mathbf{S}\mathbf{x} + \mathbf{S}^{-1}\mathbf{g}(\mathbf{S}\mathbf{x}), \tag{1.3}$$

where $\mathbf{S}^{-1}\mathbf{A}\mathbf{S}$ is a diagonal matrix having for its elements the eigenvalues of the matrix $\mathbf{A}$ arranged in the order given above. There is no need to give equations for x_3 and x_5 because these variables are complex conjugates of x_4 and x_6: $x_3 = \overline{x_4}$ and $x_5 = \overline{x_6}$. Hence, system (1.3) becomes

$$\begin{aligned}
\frac{dx_1}{d\tau} &= i(x_4^2 - x_3^2) + i\lambda(x_5^2 - x_6^2),\\
\frac{dx_2}{d\tau} &= \frac{(a-c)\xi\zeta}{ac\lambda^2} x_2(x_3 + x_4) + \frac{i}{ac\lambda^3}(\lambda^2 - 1)(x_3 + x_4)(x_5 - x_6),\\
\frac{dx_4}{d\tau} &= ix_4 + \frac{1}{2}(c-a)\xi\zeta x_2^2 + \frac{(1-\lambda)(c-a)\xi\zeta}{2ac\lambda^2} x_5^2\\
&\quad + \frac{(1+\lambda)(c-a)\xi\zeta}{2ac\lambda^2} x_6^2 + \frac{1}{2} ix_1(x_3 + x_4)\\
&\quad - \frac{1}{2} i\left[1 - \frac{c-a}{\lambda}\left(\frac{1}{c}\xi^2 - \frac{1}{a}\zeta^2\right)\right] x_2x_5\\
&\quad - \frac{1}{2} i\left[1 + \frac{c-a}{\lambda}\left(\frac{1}{c}\xi^2 - \frac{1}{a}\zeta^2\right)\right] x_2x_6 - \frac{(c-a)\xi\zeta}{ac\lambda^2} x_5x_6,\\
\frac{dx_6}{d\tau} &= i\lambda x_6 + \frac{1}{2} i\lambda x_1(x_6 - x_5)\\
&\quad + \frac{1}{2} i\left[1 - \lambda + \frac{1}{\lambda}\left(\frac{a}{c}\xi^2 + \frac{c}{a}\zeta^2\right)\right] x_2x_3\\
&\quad + \frac{1}{2} i\left[-1 - \lambda + \frac{1}{\lambda}\left(\frac{a}{c}\xi^2 + \frac{c}{a}\zeta^2\right)\right] x_2x_4\\
&\quad + \frac{(a-c)(1-\lambda)\xi\zeta}{2ac\lambda^2} x_3(x_5 - x_6) + \frac{(a-c)(1+\lambda)\xi\zeta}{2ac\lambda^2} x_4(x_5 - x_6).
\end{aligned} \tag{1.4}$$

The three known first integrals, namely, the trivial integral, that of the kinetic momentum relative to the vertical axis Oz^*, and the

energy integral, are expressed in terms of diagonal variables as follows

$$2x_1 + x_1^2 - (x_3 - x_4)^2 + (x_5 + x_6)^2 = 0, \tag{1.5}$$

$$\begin{gathered} ac\lambda^2 x_2 + ac\lambda^2 x_1 x_2 + i\,(c-a)\,\xi\zeta x_2\,(x_3 - x_4) \\ + \frac{\lambda-1}{\lambda}(x_3x_5 + x_4x_6) + \frac{\lambda+1}{\lambda}(x_3x_6 + x_4x_5) = k \end{gathered} \tag{1.6}$$

$$\left(k = \frac{k_{Oz^*}}{B\tilde{\nu}}\right),$$

$$-2x_1 + ac\lambda^2 x_2^2 + (x_3 + x_4)^2 - (x_5 - x_6)^2 = \mu^2 \tag{1.7}$$

$$\left(\mu^2 = \frac{2h}{B\tilde{\nu}^2} + 2\right).$$

1.2. Reduction to the Lyapunov form ([108a], §§ 33-45). The matrix **T** of the linear transformation of the diagonal system (1.4) to the Lyapunov skew-symmetric form is easily constructed

$$\mathbf{T} = \mathbf{I}_2 \dot{+} \mathbf{T}_2 \dot{+} \mathbf{T}_2, \quad \mathbf{I}_2 = \begin{Vmatrix} 1 & 0 \\ 0 & 1 \end{Vmatrix}, \quad \mathbf{T}_2 = \frac{1}{2}\begin{Vmatrix} 1 & -i \\ 1 & i \end{Vmatrix}.$$

The matrix **L** of the resultant transformation of the initial system (1.1) to a system similar to Lyapunov is given by

$$\mathbf{L} = \mathbf{ST},$$

where **S** is given by (1.2). The matrix and its inverse are

$$\mathbf{L} = \begin{Vmatrix} 0 & \xi & 0 & 0 & 0 & -\dfrac{\zeta}{a\lambda} \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & \zeta & 0 & 0 & 0 & \dfrac{\xi}{c\lambda} \\ \xi & 0 & 0 & -\zeta & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ \zeta & 0 & 0 & \xi & 0 & 0 \end{Vmatrix},$$

$$\mathbf{L}^{-1} = \begin{Vmatrix} 0 & 0 & 0 & \xi & 0 & \zeta \\ \dfrac{\xi}{c\lambda^2} & 0 & \dfrac{\zeta}{a\lambda^2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\zeta & 0 & \xi \\ 0 & 0 & 0 & 0 & 1 & 0 \\ -\dfrac{\zeta}{\lambda} & 0 & \dfrac{\xi}{\lambda} & 0 & 0 & 0 \end{Vmatrix}. \tag{2.1}$$

The change of the variables $\mathbf{u} = \mathbf{L}\mathbf{z}$ reduces system (1.1) to the Lyapunov form

$$\frac{dz_1}{d\tau} = -z_3z_4 + \lambda z_5z_6,$$

$$\frac{dz_2}{d\tau} = \frac{(a-c)\,\xi\zeta}{ac\lambda^2} z_2z_3 + \frac{\lambda^2-1}{ac\lambda^3} z_3z_6,$$

$$\frac{dz_3}{d\tau} = -z_4 + (c-a)\,\xi\zeta z_2^2 + \frac{(a-c)\,\xi\zeta}{ac\lambda^2} z_6^2 + \frac{c-a}{\lambda}\left(\frac{1}{c}\xi^2 - \frac{1}{a}\zeta^2\right) z_2z_6,$$

$$\frac{dz_4}{d\tau} = z_3 + z_1z_3 - z_2z_5 + \frac{(c-a)\,\xi\zeta}{ac\lambda} z_5z_6,$$

$$\frac{dz_5}{d\tau} = -\lambda z_6 - \lambda z_1z_6 + z_2z_4 + \frac{(a-c)\,\xi\zeta}{ac\lambda} z_4z_6,$$

$$\frac{dz_6}{d\tau} = \lambda z_5 + \frac{1}{\lambda}\left(\frac{a-1}{c}\xi^2 + \frac{c-1}{a}\zeta^2\right) z_2z_3 + \frac{(c-a)\,\xi\zeta}{ac\lambda^2} z_3z_6. \qquad (2.2)$$

In the Lyapunov variables, the first three integrals in the same order are

$$2z_1 + z_1^2 + z_4^2 + z_5^2 = 0, \qquad (2.3)$$

$$ac\lambda^2 z_2 + ac\lambda^2 z_1z_2 + (c-a)\,\xi\zeta z_2z_4 + \frac{1}{\lambda} z_4z_6 + z_3z_5 = k, \qquad (2.4)$$

$$-2z_1 + ac\lambda^2 z_2^2 + z_3^2 + z_6^2 = \mu^2. \qquad (2.5)$$

The simplest Chetaev linear combination of integrals ([38b], pp. 430-431) here reduces to the sum of the trivial integral (2.3) and the energy integral (2.5), and yields a positive-definite form V in all variables

$$V = z_1^2 + ac\lambda^2 z_2^2 + z_3^2 + z_4^2 + z_5^2 + z_6^2 = \mu^2.$$

This expression bears upon the stability of the lower equilibrium position and enables us to determine the stability region in the large in the variables p, q, r, γ, γ', and γ''.

1.3. Resonances. Formula (1.1a) is the ratio of the frequencies of the linear part of system (1.1). Equating this ratio to $\rho = m/n$, where m and n are mutually prime natural numbers, we obtain

$$\frac{1}{c}\xi^2 + \frac{1}{a}\zeta^2 = \rho^2, \qquad (3.1)$$

or

$$a(\rho^2c - 1)\,\xi^2 + c(\rho^2a - 1)\,\zeta^2 = 0 \quad (\xi^2 + \zeta^2 \equiv 1). \qquad (3.1a)$$

In the first octant of space $\{a, c, \xi^2\}$ equation (3.1) describes a portion of a second-order surface

$$(a-c)\,\xi^2 = c(\rho^2a - 1) \qquad (3.1b)$$

with the natural constraints

$$0 \leqslant \xi^2 \leqslant 1, \quad a + c \geqslant 1, \quad c - a \leqslant 1, \quad a - c \leqslant 1. \tag{3.2}$$

The last three inequalities represent a well-known relationship, namely, that the sum of any two principal moments of inertia is not smaller than the third. When $\rho = 1$, equation (3.1) becomes

$$A\,(C - B)\,\xi^2 + C\,(A - B)\,\zeta^2 = 0. \tag{3.3}$$

Zelman [398b] noted that these are the conditions imposed on the parameters in the Gess-Appelrot case [213]. Equation (3.3) is evidently satisfied in the Lagrange-Poisson case as well ($\xi = 0, A = B$).

When $\rho = 2$, equation (3.1) becomes

$$A\,(4C - B)\,\xi^2 + C\,(4A - B)\,\zeta^2 = 0.$$

If $A = B$, $\zeta = 0$, then $A = 4C$, and this, as Zelman remarked [398b], characterizes the parameters of a solid body in the Goryachev-Chaplygin case [65b, 88].

However, equation (3.1) (or (3.1a), (3.1b)), which has a solution for any $0 < \rho = m/n < \infty$, has no solution in the Kovalevskaya case [92] ($\rho = \sqrt{2}$).

1.4. Simplest motions. We begin with permanent rotations (those with constant p, q, r, γ, γ', and γ'', that is, with fixed orientation of the body's axis of rotation and with constant angular velocity). Equations (1.1) yield

$$\begin{aligned} (1 - c)\,QR - \zeta\gamma' &= 0, \quad & R\gamma' - Q\gamma'' &= 0, \\ (c - a)\,RP + \zeta\gamma - \xi\gamma'' &= 0, \quad & P\gamma'' - R\gamma &= 0, \\ (a - 1)\,PQ + \xi\gamma' &= 0, \quad & Q\gamma - P\gamma' &= 0. \end{aligned} \tag{4.1}$$

The last three equations yield

$$P = \Omega\gamma, \quad Q = \Omega\gamma', \quad R = \Omega\gamma'' \quad (\Omega = +\sqrt{P^2 + Q^2 + R^2}), \tag{4.2}$$

which states that the permanent axis is always vertical. In order to find its position in the body, we multiply the first and third equations of (4.1) by ξ and ζ, respectively, and add them; by (4.2), we obtain

$$\gamma'\,[\xi\,(1 - c)\,\gamma'' + \zeta\,(a - 1)\,\gamma] = 0.$$

The locus of axes of permanent rotation in the body (the Staude cone) is divided, in the case in question, into two planes

$$y = 0 \quad \text{and} \quad \xi\,(1 - c)\,z + \zeta\,(a - 1)\,x = 0.$$

Now we consider pendulum oscillations. Such motions are known to be possible with respect to the principal axis of inertia (it is necessarily horizontal and fixed in space) that is perpendicular to the plane containing the centroid. In fact, the statement of the problem

was determined by the existence of pendulum oscillations. Hence, we set $P = R = \Gamma' = 0$ in equations (1.1) and, returning to the variables γ and γ'', we obtain

$$\frac{dQ}{d\tau} = \zeta\gamma - \xi\gamma'', \quad \frac{d\gamma}{d\tau} = -Q\gamma'', \quad \frac{d\gamma''}{d\tau} = Q\gamma. \tag{4.3}$$

From $\gamma'' = \cos\vartheta$, where ϑ is the nutation angle, we derive that $\gamma = \sin\vartheta$ for $\gamma' \equiv 0$. In the equilibrium position $\cos\vartheta = \zeta$, so that for the pendulum motion we set

$$\vartheta = \arccos\zeta + \Theta.$$

Equations (4.3) now yield an equation for the pendulum oscillations

$$\frac{d^2\Theta}{d\tau^2} + \sin\Theta = 0.$$

1.5. Transformation of equations of diagonal form. The trivial integral (1.5) yields for x_1

$$x_1 = \frac{1}{2}(x_3 - x_4)^2 - \frac{1}{2}(x_5 + x_6)^2 + (4). \tag{5.1}$$

Note that x_2 and k are of the same order of smallness; therefore, substituting (5.1) into the kinetic momentum integral (1.6), we arrive at

$$x_2 = \frac{1}{ac\lambda^2}k + \frac{1-\lambda}{ac\lambda^3}(x_3x_5 + x_4x_6)$$
$$- \frac{1+\lambda}{ac\lambda^3}(x_3x_6 + x_4x_5) + \frac{i(a-c)\xi\zeta}{a^2c^2\lambda^4}k(x_3 - x_4) + (3). \tag{5.2}$$

Here and henceforth we indicate the order of smallness of the omitted terms in the variables x_3, x_4, x_5, x_6, and in the constant k. The order of system (1.4) is now lowered by two, and the transformed system consisting of two pairs of complex conjugate equations ($x_3 = \bar{x}_4$ and $x_5 = \bar{x}_6$) reduces to

$$\frac{dx_4}{d\tau} = ix_4 + \frac{(c-a)\xi\zeta}{2a^2c^2\lambda^4}k^2 + \frac{(1-\lambda)(c-a)\xi\zeta}{2ac\lambda^2}x_5^2$$
$$+ \frac{(1+\lambda)(c-a)\xi\zeta}{2ac\lambda^2}x_6^2 - \frac{i}{2ac\lambda^2}\left[1 - \frac{c-a}{\lambda}\left(\frac{1}{c}\xi^2 - \frac{1}{a}\zeta^2\right)\right]kx_5$$
$$- \frac{i}{2ac\lambda^2}\left[1 + \frac{c-a}{\lambda}\left(\frac{1}{c}\xi^2 - \frac{1}{a}\zeta^2\right)\right]kx_6 - \frac{(c-a)\xi\zeta}{ac\lambda^2}x_5x_6$$
$$+ \frac{1}{4}i(x_3 - x_4)(x_3^2 - x_4^2) - \frac{1}{4}i(x_3 + x_4)(x_5 + x_6)^2$$
$$- \frac{(c-a)(1+\lambda)\xi\zeta}{a^2c^2\lambda^5}k(x_3x_6 + x_4x_5)$$
$$+ \frac{(c-a)(1-\lambda)\xi\zeta}{a^2c^2\lambda^5}k(x_3x_5 + x_4x_6) - i\frac{(a-c)^2\xi^2\zeta^2}{a^3c^3\lambda^6}k^2(x_3 - x_4)$$

$$+\frac{1}{2ac\lambda^3}\Big[i(\lambda+1)(x_3x_6+x_4x_5)+i(\lambda-1)(x_3x_5+x_4x_6)$$

$$+\frac{(a-c)\xi\zeta}{ac\lambda}k(x_3-x_4)\Big]\Big\{\Big[1-\frac{c-a}{\lambda}\Big(\frac{1}{c}\xi^2-\frac{1}{a}\zeta^2\Big)\Big]x_5$$

$$+\Big[1+\frac{c-a}{\lambda}\Big(\frac{1}{c}\xi^2-\frac{1}{a}\zeta^2\Big)\Big]x_6\Big\}+(4),$$

$$\frac{dx_6}{d\tau}=i\lambda x_6+\frac{i}{2ac\lambda^2}\Big[1-\lambda+\frac{1}{\lambda}\Big(\frac{a}{c}\xi^2+\frac{c}{a}\zeta^2\Big)\Big]kx_3$$

$$+\frac{(a-c)(1-\lambda)\xi\zeta}{2ac\lambda^2}x_3(x_5-x_6)+\frac{i}{2ac\lambda^2}\Big[-1-\lambda$$

$$+\frac{1}{\lambda}\Big(\frac{a}{c}\xi^2+\frac{c}{a}\zeta^2\Big)\Big]kx_4+\frac{(a-c)(1+\lambda)\xi\zeta}{2ac\lambda^2}x_4(x_5-x_6)$$

$$+\frac{1}{4}i\lambda(x_3-x_4)^2(x_6-x_5)+\frac{1}{4}i\lambda(x_5^2-x_6^2)(x_5+x_6)$$

$$+\frac{1}{2ac\lambda^3}\Big[i(1-\lambda)(x_3x_5+x_4x_6)-i(1+\lambda)(x_3x_6+x_4x_5)$$

$$+\frac{(c-a)\xi\zeta}{ac\lambda}k(x_3-x_4)\Big]\Big\{\Big[1-\lambda+\frac{1}{\lambda}\Big(\frac{a}{c}\xi^2+\frac{c}{a}\zeta^2\Big)\Big]x_3$$

$$+\Big[-1-\lambda+\frac{1}{\lambda}\Big(\frac{a}{c}\xi^2+\frac{c}{a}\zeta^2\Big)\Big]x_4\Big\}+(4). \qquad (5.3)$$

The remaining first integral is best approached as a sum of the energy integral (1.7) and the trivial integral (1.5). Using (5.1) and (5.2), we can write the first integral of system (5.3) as

$$\frac{1}{ac\lambda^2}\Big[k+\frac{1-\lambda}{\lambda}(x_3x_5+x_4x_6)-\frac{1+\lambda}{\lambda}(x_3x_6+x_4x_5)$$

$$+i\frac{(a-c)\xi\zeta}{ac\lambda^2}k(x_3-x_4)\Big]^2+4\,|\,x_4\,|^2+4\,|\,x_6\,|^2+(4)=\mu^2.$$

1.6. Possible generalizations. The system of differential equations analyzed above is three-parametric owing to the number of independent parameters involved: a, c, and ξ (we recall that $\zeta = +\sqrt{1-\xi^2}$). The system would be reduced to two-parametric if the ellipsoid of inertia with respect to point O were an ellipsoid of revolution ($a = 1$ or $c = 1$ or $a = c$), or if the centroid were on one of the principal axes of an arbitrary ellipsoid of inertia with respect to point O ($\xi = 0$ or $\xi = 1$). In the Lagrange-Poisson case ($a = 1$, $\xi = 0$), system (1.1) has a single parameter c. In the case of kinetic symmetry ($a = c = 1$, $\xi = 0$) and in the Kovalevskaya case $\left(a = 1,\ c = \frac{1}{2},\ \xi = 1 \text{ or } a = c = 2,\ \xi = 1\right)$, system (1.1) has no parameters. The motion of a solid body in the general case is described by a system of differential equations with four parameters: a, c, ξ, and η ($\zeta = +\sqrt{1-\xi^2-\eta^2}$).

The four-parameter equations are similar to those given above but are more cumbersome (simpler equations are obtained with another approach outlined in Section 2). The equations of oscillations about the equilibrium position in other force fields (such as a central Newton force field) are also unwieldy.

The transformations outlined above enable us to apply methods specific to oscillations in essentially nonlinear autonomous systems to the motion of a heavy solid body. Some of these methods were presented in Chapters I, IV, and VIII. The future will show whether these methods are really effective for analyzing oscillations of solids. The next subsection discusses a method that appears effective to the author.

1.7. Situation similar to the Kovalevskaya case. Within the assumptions of the present section, we consider a situation when the centroid G of a body is in the equatorial plane of the ellipsoid of inertia (for the fixed point O), which is also the ellipsoid of revolution. We thus assume

$$A = B \neq C, \quad y_G = z_G = 0, \quad x_G = OG \neq 0.$$

This situation covers the Kovalevskaya case as well $\left(C=\frac{1}{2}A\right)$.

For $a=1$, $\xi=1$, and $\zeta=0$ equations (1.1) become

$$\frac{dP}{d\tau}=(1-c)\,QR, \quad \frac{dQ}{d\tau}=-\gamma''+(c-1)\,RP, \quad \frac{dR}{d\tau}=\frac{1}{c}\gamma',$$
$$\frac{d\gamma}{d\tau}=R\gamma'-Q\gamma'', \quad \frac{d\gamma'}{d\tau}=P\gamma''-R\gamma, \quad \frac{d\gamma''}{d\tau}=Q\gamma-P\gamma'. \tag{7.1}$$

The first three integrals, namely, the trivial, kinetic momentum with respect to Oz_1, and energy integrals, take the form

$$\gamma^2+\gamma'^2+\gamma''^2=1, \quad P\gamma+Q\gamma'+cR\gamma''=k \quad \left(k=\frac{1}{\tilde{\nu}A}K_{z_1}\right), \tag{7.2}$$

$$P^2+Q^2+cR^2-2\gamma=\frac{2h}{A\tilde{\nu}^2}. \tag{7.3}$$

In order to investigate oscillations about the lower equilibrium position ($\gamma = 1$, $\gamma' = \gamma'' = 0$), we solve the first two integrals for γ and P

$$\gamma=+\sqrt{1-\gamma'^2-\gamma''^2}, \quad P=\frac{k-Q\gamma'-cR\gamma''}{\sqrt{1-\gamma'^2-\gamma''^2}}. \tag{7.4}$$

The last expressions are analytic functions of the variables if

$$\gamma'^2+\gamma''^2<1, \tag{7.5}$$

that is, until the centroid G reaches the horizontal plane passing through O. We assume that the centroid of the body reaches this plane at zero angular velocity, that is,

$$P = Q = R = \gamma = 0, \quad \gamma'^2 + \gamma''^2 = 1.$$

The constant h in integral (7.3) equals zero for this motion; at the initial moment, by (7.3), we have

$$P^2(0) + Q^2(0) + cR^2(0) - 2\gamma(0) = 0.$$

Therefore, the centroid will never reach the horizontal plane in question if at the initial moment

$$P^2(0) + Q^2(0) + cR^2(0) - 2\gamma(0) < 0. \tag{7.6}$$

In other words, the condition of analyticity (7.5) is satisfied.

Substitution of (7.4) into (7.1) yields a fourth-order system

$$\frac{dQ}{d\tau} = -\gamma'' + (c-1)R\frac{k - Q\gamma' - cR\gamma''}{\sqrt{1-\gamma'^2-\gamma''^2}}, \quad \frac{dR}{d\tau} = \frac{1}{c}\gamma',$$

$$\frac{d\gamma'}{d\tau} = \gamma''\frac{k - Q\gamma' - cR\gamma''}{\sqrt{1-\gamma'^2-\gamma''^2}} - R\sqrt{1-\gamma'^2-\gamma''^2},$$

$$\frac{d\gamma''}{d\tau} = Q\sqrt{1-\gamma'^2-\gamma''^2} - \gamma'\frac{k - Q\gamma' - cR\gamma''}{\sqrt{1-\gamma'^2-\gamma''^2}} \tag{7.7}$$

that is analytic if (7.6) is satisfied.

The characteristic equation for the linear part of system (7.7) is

$$\lambda^4 + \left(1 + \frac{1}{c} + k^2\right)\lambda^2 + \frac{1}{c} + \frac{1-c}{c}k^2 = 0.$$

The roots of this equation are pure imaginary and distinct if $c < 1$. Let $c > 1$; then one positive root appears if

$$k^2 > \frac{1}{c-1}. \tag{7.8}$$

Note that the boundary given by (7.8) for the instability in the large of the lower equilibrium position is arbitrarily high for a thick disk ($c = 1 + \varepsilon$, $\varepsilon > 0$) and arbitrarily low for a thin disk ($c \gg 1$).

Let us analyze the compatibility of condition (7.8) (instability in the large) and (7.6) (analyticity), taking into account that by (7.2)

$$k = P(0)\gamma(0) + Q(0)\gamma'(0) + cR(0)\gamma''(0).$$

Let the motion start from equilibrium, that is,

$$\gamma(0) = 1, \quad \gamma'(0) = \gamma''(0) = 0.$$

Then $k = P(0)$, and conditions (7.8) and (7.6) can be written as the inequalities

$$\frac{1}{c-1} < P^2(0) < 2 - Q^2(0) - R^2(0), \tag{7.9}$$

which are compatible only when $c > \frac{3}{2}$.

1.8. Application of the method of successive approximations. We recast system (7.7) in vector form

$$\frac{d\mathbf{x}}{d\tau} - \mathbf{A}\mathbf{x} = \mathbf{f}(\mathbf{x}), \quad \mathbf{x} = \begin{pmatrix} Q \\ R \\ \gamma' \\ \gamma'' \end{pmatrix}, \quad \mathbf{A} = \begin{Vmatrix} 0 & (c-1)k & 0 & -1 \\ 0 & 0 & \frac{1}{c} & 0 \\ 0 & -1 & 0 & k \\ 1 & 0 & -k & 0 \end{Vmatrix}, \tag{8.1}$$

where $\mathbf{f}(\mathbf{x})$ is a vector-function composed of nonlinear terms. We select $\mathbf{x}_0 = \mathbf{x}(0)$ as the zeroth approximation; consequently, in the first approximation

$$\frac{d\mathbf{x}_1}{d\tau} - \mathbf{A}\mathbf{x}_1 = \mathbf{f}(\mathbf{x}_0) \tag{8.2}$$

and in higher approximations

$$\frac{d\mathbf{x}_{k+1}}{d\tau} - \mathbf{A}\mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k) \quad (k = 1, 2, \ldots),$$

whence

$$\mathbf{x}_{k+1} = e^{\tau\mathbf{A}}\,\mathbf{x}(0) + \int_0^\tau e^{(\tau-s)\mathbf{A}}\mathbf{f}(\mathbf{x}_k(s))\,ds. \tag{8.3}$$

Let us evaluate the norm of the difference in the standard manner

$$\mathbf{x}_{k+1}(\tau) - \mathbf{x}_k(\tau) = \int_0^\tau e^{(\tau-s)\mathbf{A}}\,[\mathbf{f}(\mathbf{x}_k(s)) - \mathbf{f}(\mathbf{x}_{k-1}(s))]\,ds \quad (k = 1, 2, \ldots).$$

This gives

$$|\,\mathbf{x}_{k+1}(\tau) - \mathbf{x}_k(\tau)\,| \leqslant 4^{3/2}L\tau\,|\,e^{(\tau-s)\mathbf{A}}\,|\,|\,\mathbf{x}_k(s) - \mathbf{x}_{k-1}(s)\,|$$
$$(0 \leqslant s \leqslant \tau; \quad k = 1, 2, \ldots),$$

where L is the Lipschitz constant in the closed domain contained in (7.5). Therefore, the mapping (8.3) is contracting and the sequence $\{\mathbf{x}_k(\tau)\}$ is uniformly convergent in the segment $0 \leqslant \tau \leqslant \tau^*$ if

$$\tau < \tau^* = \frac{1}{8L}\,|\,e^{(\tau-s)\mathbf{A}}\,|^{-1}.$$

Equation (8.2) and formula (8.3) yield

$$\mathbf{x}_1(\tau) = e^{\tau \mathbf{A}} \mathbf{x}(0) + \mathbf{A}^{-1}(e^{\tau \mathbf{A}} - \mathbf{I}_4)\, \mathbf{f}(\mathbf{x}_0) \tag{8.4}$$

for the first approximation. In detailed form, formula (8.4) describes the "coin effect", that is, the standing of a thin disk spun at a sufficiently high angular velocity around a vertical axis.

1.9. Remarks on the determination of the position of a solid body with a fixed point. The approach used above was integration of the Euler-Poisson equations. Only five arbitrary constants are obtained, however, after p, q, r, γ, γ', and γ'' are found for a given time t, since at any moment the sum of squared direction cosines equals unity. At the same time, the initial data can assume six arbitrary initial values, for example, p_0, q_0, r_0, φ_0, ψ_0, and ϑ_0, where φ, ψ, and ϑ are Euler's angles. It can readily be shown (see, for instance, [65b], Ch. I, § 5) that in addition to integration of the Euler-Poisson equations, one additional integration is required to obtain the complete solution. Indeed, if the formulas for φ and ϑ are [65b]

$$\varphi = \arctan \frac{\gamma}{\gamma'}, \quad \vartheta = \arccos \gamma'',$$

the precession angle ψ is found from

$$\frac{d\psi}{dt} = \frac{1}{\gamma'} \left[r - \frac{1}{\gamma'^2 + \gamma''^2} \left(\frac{d\gamma}{dt} \gamma' - \frac{d\gamma'}{dt} \gamma \right) \right].$$

Kharlamov uses a different method of finding the body's position in space, namely, by means of a fixed and a mobile hodographs of angular velocity and arc coordinates. By analogy with kinematics of a point, we refer to this method ([88], Sec. 1.6) as a natural one.

§ 2. The General Case

As Ishlinskii remarks ([79a], Ch. IV, § 1), the system of Euler's dynamic and kinematic equations is not very convenient for gyroscopic analysis. This remark is equally valid for the problem discussed in this chapter. In the general case, if the Euler-Poisson equations are used, there are four independent parameters: two ratios of moments of inertia and two ratios of the centroid's coordinates to the distance from the fixed point. The number of independent parameters is less than four if, as in Section 1, the centroid lies in one of the principal planes of the ellipsoid of inertia for the fixed point (this is always possible if the ellipsoid is one of revolution). The assumptions made above reduce the number of independent parameters to three; in the case described of the ellipsoid of revolution the number is not greater than two; it equals unity in the Lagrange-Poisson case; and, finally, no parameters are involved in the Kovalevskaya case.

If the Euler-Poisson equations are used, transformations to the Jordan form of the linear part of the equations in question are very cumbersome, but they contract considerably if the number of the parameters is at least one less than in the general case (see Subsection 1.1). The manipulations are substantially simplified, however, if one uses the special axes of the frame of reference inherent in the body, introduced by Kharlamov ([88], Sec. 2.6).

Both the preceding section and this one are devoted to preliminary transformations necessary before applying methods of small parameter and the method of normal forms. The actual application of these methods, however, requires considerable effort. This reflects an essential point: in general, the motion of a solid body with a fixed point cannot be reduced to a superimposition of oscillations, but has to be described as a slip-free rolling motion of a moving axoid over a fixed one (a detailed analysis can be found in [88]). As for the method of successive approximations, it appears that the results given in Subsection 2.6 (and also 1.8) are sufficiently effective.

2.1. Base reference frame. We assume that the centroid G of a body does not coincide with the fixed point O. We draw the first

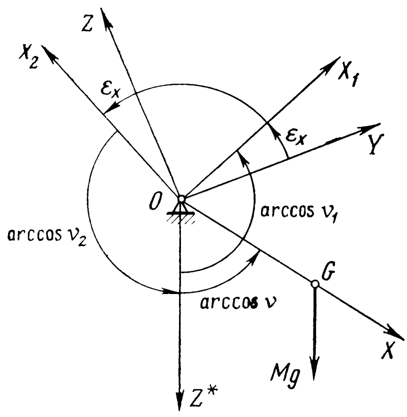

FIG. 19

axis, OX, through the centroid G and choose the axes OY and OZ such that the centrifugal moment of inertia J_{YZ} equals zero. The reference frame $OXYZ$ inherent in the body is called the base reference frame (Fig. 19).

The kinetic energy of a body is a quadratic form of the components of the angular velocity

$$K=\frac{1}{2}(J_{XX}\omega_X^2+J_{YY}\omega_Y^2+J_{ZZ}\omega_Z^2+2J_{XY}\omega_X\omega_Y+2J_{ZX}\omega_Z\omega_X), \quad (1.1)$$

where J_{XX}, J_{YY}, and J_{ZZ} are the axial and $J_{XY} = J_{YX}$ and $J_{ZX} = J_{XZ}$ ($J_{YZ} = J_{ZY} = 0$) are the centrifugal moments of inertia of the body. The formulas for the projections of the kinetic momentum $\mathbf{k}_O$ of the body onto the base frame axes,

$$
\begin{aligned}
k_X &= J_{XX}\omega_X + J_{XY}\omega_Y + J_{XZ}\omega_Z,\\
k_Y &= J_{YX}\omega_X + J_{YY}\omega_Y,\\
k_Z &= J_{ZX}\omega_X + J_{ZZ}\omega_Z,
\end{aligned}
\tag{1.2}
$$

enable us to rewrite (1.1) in the form

$$K = \frac{1}{2}(k_X\omega_X + k_Y\omega_Y + k_Z\omega_Z). \tag{1.3}$$

We solve equations (1.2) with respect to ω_X, ω_Y, and ω_Z

$$
\begin{aligned}
\omega_X &= \Omega_{XX}k_X + \Omega_{XY}k_Y + \Omega_{XZ}k_Z,\\
\omega_Y &= \Omega_{YX}k_X + \Omega_{YY}k_Y + \Omega_{YZ}k_Z,\\
\omega_Z &= \Omega_{ZX}k_X + \Omega_{ZY}k_Y + \Omega_{ZZ}k_Z,
\end{aligned}
\tag{1.4}
$$

where

$$\Omega_{XX} = \frac{J_{YY}J_{ZZ}}{\det \mathbf{J}}, \quad \Omega_{YY} = \frac{J_{ZZ}J_{XX} - J_{ZX}^2}{\det \mathbf{J}}, \quad \Omega_{ZZ} = \frac{J_{XX}J_{YY} - J_{XY}^2}{\det \mathbf{J}},$$

$$\Omega_{XY} = \Omega_{YX} = -\frac{J_{XY}J_{ZZ}}{\det \mathbf{J}}, \quad \Omega_{XZ} = \Omega_{ZX} = -\frac{J_{XZ}J_{YY}}{\det \mathbf{J}},$$

$$\Omega_{YZ} = \Omega_{ZY} = \frac{J_{XY}J_{ZX}}{\det \mathbf{J}},$$

and, finally, $\det \mathbf{J}$ is the determinant of the positive-definite quadratic form (1.1), that is, of the matrix

$$\mathbf{J} = \left\| \begin{matrix} J_{XX} & J_{XY} & J_{ZX} \\ J_{XY} & J_{YY} & 0 \\ J_{ZX} & 0 & J_{ZZ} \end{matrix} \right\|. \tag{1.5}$$

Substitution of (1.4) into (1.3) yields an expression for the kinetic energy of the body as a quadratic form of the component of the kinetic momentum with respect to the fixed point O

$$
\begin{aligned}
K = \frac{1}{2}(&\Omega_{XX}k_X^2 + \Omega_{YY}k_Y^2 + \Omega_{ZZ}k_Z^2\\
&+ 2\Omega_{XY}k_Xk_Y + 2\Omega_{YZ}k_Yk_Z + 2\Omega_{ZX}k_Zk_X).
\end{aligned}
\tag{1.6}
$$

This completes the transformation from the inertia tensor $\{J_{XX}, \ldots, J_{ZX}\}$ to the gyration tensor $\{\Omega_{XX}, \ldots, \Omega_{ZX}\}$ (see [88], Sec. 2.5).

In a more compact form, this becomes

$$\mathbf{k}_O = \mathbf{J}\omega, \qquad K = \frac{1}{2}(\mathbf{k}_O,\ \omega) = \frac{1}{2}(\mathbf{J}\omega,\ \omega),$$

$$\omega = \mathbf{J}^{-1}\mathbf{k}_O, \quad K = \frac{1}{2}(\mathbf{k}_O,\ \mathbf{J}^{-1}\mathbf{k}_O). \tag{1.7}$$

2.2. Special reference frame. Let us rotate the axes of the base reference frame $OXYZ$ about OX by an angle ε, moving OY in the direction of OZ; the new axes will be denoted by OXX_1X_2 (see Fig. 19). The new unit vectors are expressed in terms of those of the base reference frame by the equations

$$\mathbf{e}_{X_1} = \mathbf{e}_Y \cos\varepsilon + \mathbf{e}_Z \sin\varepsilon, \ \ \mathbf{e}_{X_2} = -\mathbf{e}_Y \sin\varepsilon + \mathbf{e}_Z \cos\varepsilon.$$

The matrix $\mathbf{R}$ composed of the components of the new base frame $\mathbf{e}_X$, $\mathbf{e}_{X_1}$, and $\mathbf{e}_{X_2}$ is given in the base frame $\mathbf{e}_X$, $\mathbf{e}_Y$, and $\mathbf{e}_Z$ by

$$\mathbf{R} = \begin{Vmatrix} 1 & 0 & 0 \\ 0 & \cos\varepsilon & -\sin\varepsilon \\ 0 & \sin\varepsilon & \cos\varepsilon \end{Vmatrix}.$$

This matrix is orthogonal: $\mathbf{R}^{-1} = \mathbf{R}^{\tau}$.

The matrix $\tilde{\mathbf{J}}$ of the inertia tensor is expressed in the new base frame by the formula (see (I.1.27), [80])

$$\tilde{\mathbf{J}} = \mathbf{R}^{-1}\mathbf{J}\mathbf{R}$$

$$= \begin{Vmatrix} J_{XX} & J_{XY}\cos\varepsilon + J_{ZX}\sin\varepsilon & J_{ZX}\cos\varepsilon - J_{XY}\sin\varepsilon \\ J_{XY}\cos\varepsilon + J_{ZX}\sin\varepsilon & J_{YY}\cos^2\varepsilon + J_{ZZ}\sin^2\varepsilon & \frac{1}{2}(J_{ZZ} - J_{YY})\sin 2\varepsilon \\ J_{ZX}\cos\varepsilon - J_{XY}\sin\varepsilon & \frac{1}{2}(J_{ZZ} - J_{YY})\sin 2\varepsilon & J_{YY}\sin^2\varepsilon + J_{ZZ}\cos^2\varepsilon \end{Vmatrix}.$$

Obviously, $\det\tilde{\mathbf{J}} = \det\mathbf{J}$. For the elements of the inverse matrix we derive

$$\tilde{J}_{11}^{-1} = \frac{J_{YY}J_{ZZ}}{\det\mathbf{J}}, \quad \tilde{J}_{22}^{-1}(\varepsilon) = \frac{1}{\det\mathbf{J}}\left[(J_{XX}J_{YY} - J_{XY}^2)\sin^2\varepsilon \right.$$

$$\left. + (J_{XX}J_{ZZ} - J_{ZX}^2)\cos^2\varepsilon + J_{XY}J_{ZX}\sin 2\varepsilon\right],$$

$$\tilde{J}_{33}^{-1}(\varepsilon) = \frac{1}{\det\mathbf{J}}\left[(J_{XX}J_{ZZ} - J_{ZX}^2)\sin^2\right.$$

$$\left. + (J_{XX}J_{YY} - J_{XY}^2)\cos^2\varepsilon - J_{XY}J_{ZX}\sin 2\varepsilon\right],$$

$$\tilde{J}_{12}^{-1}(\varepsilon) = \tilde{J}_{21}^{-1}(\varepsilon) = -\frac{1}{\det\mathbf{J}}(J_{YY}J_{ZX}\sin\varepsilon + J_{ZZ}J_{XY}\cos\varepsilon),$$

$$\tilde{J}_{13}^{-1}(\varepsilon) = \tilde{J}_{31}^{-1}(\varepsilon) = \frac{1}{\det\mathbf{J}}(J_{ZZ}J_{XY}\sin\varepsilon - J_{YY}J_{ZX}\cos\varepsilon),$$

$$\tilde{J}_{23}^{-1}(\varepsilon)=\tilde{J}_{32}^{-1}(\varepsilon)=\frac{1}{\det \mathbf{J}}\left[\frac{1}{2}(J_{XX}J_{YY}-J_{XY}^2-J_{XX}J_{ZZ}+J_{ZX}^2)\sin 2\varepsilon\right.$$
$$\left.+J_{XY}J_{ZX}\cos 2\varepsilon\right]. \qquad (2.1)$$

The special axes introduced by Kharlamov ([88], Sec. 2.6) satisfy the condition $\tilde{J}_{23}^{-1}(\varepsilon)=0$ imposed on the reference frame fixed to the body; therefore, the angle ε_X by which the base axes $OXYZ$ are rotated toward the special axes OXX_1X_2 (the angle is measured counterclockwise if observed from the point G toward the point O) is given by

$$\tan 2\varepsilon_X=\frac{2J_{XY}J_{ZX}}{J_{XX}J_{ZZ}-J_{ZX}^2-J_{XX}J_{YY}+J_{XY}^2}. \qquad (2.2)$$

Using Kharlamov's notation, we denote the quantities in (2.1) for $\varepsilon=\varepsilon_X$ by

$$a=\tilde{J}_{11}^{-1}=\frac{J_{YY}J_{ZZ}}{\det \mathbf{J}},\quad a_1=\tilde{J}_{22}^{-1}(\varepsilon_X),\quad a_2=\tilde{J}_{33}^{-1}(\varepsilon_X),$$
$$b_1=\tilde{J}_{12}^{-1}(\varepsilon_X),\quad b_2=\tilde{J}_{13}^{-1}(\varepsilon_X). \qquad (2.3)$$

Note that the parameters a, a_1, and a_2 are automatically positive.

Formulas (1.7) express the kinetic energy of the solid body in terms of the projections k_X, k_{X_1}, and k_{X_2} of the kinetic momentum (with respect to the fixed point) onto the special axes OXX_1X_2, so that

$$K=\frac{1}{2}(\mathbf{k}_O,\ \mathbf{J}^{-1}\mathbf{k}_O)=\frac{1}{2}(ak_X^2+a_1k_{X_1}^2+a_2k_{X_2}^2)+(b_1k_{X_1}+b_2k_{X_2})k_X. \qquad (2.4)$$

The projections of the angular velocity ω onto the special axes are

$$\omega_X=ak_X+b_1k_{X_1}+b_2k_{X_2},$$
$$\omega_{X_1}=b_1k_X+a_1k_{X_1},\quad \omega_{X_2}=b_2k_X+a_2k_{X_2}. \qquad (2.5)$$

2.3. Equations of motion of a heavy solid body in the special reference frame. The motion of a heavy solid body with a fixed point is given by the system of equations

$$\frac{d\mathbf{k}_O}{dt}=[\mathbf{k}_O\times\omega]+[l\mathbf{e}_X\times Mg\mathbf{v}^0],\ \frac{d\mathbf{v}^0}{dt}=\mathbf{v}^0\times\omega, \qquad (3.1)$$

which possesses the first integrals

$$(\mathbf{v}^0,\ \mathbf{v}^0)=1,\quad (\mathbf{k}_O,\ \mathbf{v}^0)=k_{OZ*},\quad K-(Mg\mathbf{v}^0,\ l\mathbf{e}_X)=h, \qquad (3.2)$$

where $\mathbf{k}_O$ is the kinetic momentum of the solid body with respect to the fixed point O, $\mathbf{e}_X$ is a unit vector directed toward the centroid G, ν^0 is a unit vector of gravity, and $l = OG$. As in the preceding subsection, we denote the projections of $\mathbf{k}_O$ onto the special axes OXX_1X_2 by k_X, k_{X_1}, and k_{X_2}, and direction cosines of vector ν^0 in the same reference frame by ν, ν_1, and ν_2, after which equations (3.1) and integrals (3.2) can be written (by 2.5) as

$$\frac{dk_X}{dt} = (b_2k_X + a_2k_{X_2})\,k_{X_1} - (b_1k_X + a_1k_{X_1})\,k_{X_2},$$

$$\frac{dk_{X_1}}{dt} = -Mgl\nu_2 + (ak_X + b_1k_{X_1} + b_2k_{X_2})\,k_{X_2} - (b_2k_X + a_2k_{X_2})\,k_X,$$

$$\frac{dk_{X_2}}{dt} = Mgl\nu_1 - (ak_X + b_1k_{X_1} + b_2k_{X_2})\,k_{X_1} + (b_1k_X + a_1k_{X_1})\,k_X,$$

$$\frac{d\nu}{dt} = (b_2k_X + a_2k_{X_2})\,\nu_1 - (b_1k_X + a_1k_{X_1})\,\nu_2,$$

$$\frac{d\nu_1}{dt} = (ak_X + b_1k_{X_1} + b_2k_{X_2})\,\nu_2 - (b_2k_X + a_2k_{X_2})\,\nu,$$

$$\frac{d\nu_2}{dt} = -(ak_X + b_1k_{X_1} + b_2k_{X_2})\,\nu_1 + (b_1k_X + a_1k_{X_1})\,\nu; \tag{3.3}$$

$$\nu^2 + \nu_1^2 + \nu_2^2 = 1, \quad k_X\nu + k_{X_1}\nu_1 + k_{X_2}\nu_2 = k_{OZ*},$$

$$\frac{1}{2}(ak_X^2 + a_1k_{X_1}^2 + a_2k_{X_2}^2) + (b_1k_{X_1} + b_2k_{X_2})\,k_X - Mgl\nu = h \tag{3.4}$$

(see [88], equations (2.6.8), (3.2.11), (3.2.12). (3.3.11), (3.3.16), and (3.3.18)).

One solution of system (3.3) is $k_X = k_{X_1} = k_{X_2} = 0$, $\nu_1 = \nu_2 = 0$, $\nu = 1$, which corresponds to the lower equilibrium position. Now we assume that at every moment during the oscillations

$$\nu = 1 + \mathrm{N}, \tag{3.5}$$

and introduce the following dimensionless parameters and variables

$$\tau = \sqrt{Mgla_1}\,t, \qquad \varkappa = \sqrt{\frac{a_1}{Mgl}}\,k_X,$$

$$\varkappa_1 = \sqrt{\frac{a_1}{Mgl}}\,k_{X_1}, \qquad \varkappa_2 = \sqrt{\frac{a_1}{Mgl}}\,k_{X_2},$$

$$d' = \frac{a}{a_1}, \quad d_2 = \frac{a_2}{a_1}, \quad e_1 = \frac{b_1}{a_1}, \quad e_2 = \frac{b_2}{a_1}. \tag{3.6}$$

System (3.3) and integrals (3.4) become

$$\frac{d\varkappa}{d\tau}=(e_2\varkappa+d_2\varkappa_2)\,\varkappa_1-(e_1\varkappa+\varkappa_1)\,\varkappa_2,$$
$$\frac{d\varkappa_1}{d\tau}=-\nu_2+(d'\varkappa+e_1\varkappa_1+e_2\varkappa_2)\,\varkappa_2-(e_2\varkappa+d_2\varkappa_2)\,\varkappa,$$
$$\frac{d\varkappa_2}{d\tau}=\nu_1-(d'\varkappa+e_1\varkappa_1+e_2\varkappa_2)\,\varkappa_1+(e_1\varkappa+\varkappa_1)\,\varkappa,$$
$$\frac{dN}{d\tau}=(e_2\varkappa+d_2\varkappa_2)\,\nu_1-(e_1\varkappa+\varkappa_1)\,\nu_2,$$
$$\frac{d\nu_1}{d\tau}=-e_2\varkappa-d_2\varkappa_2+(d'\varkappa+e_1\varkappa_1+e_2\varkappa_2)\,\nu_2-(e_2\varkappa+d_2\varkappa_2)\,N,$$
$$\frac{d\nu_2}{d\tau}=e_1\varkappa+\varkappa_1-(d'\varkappa+e_1\varkappa_1+e_2\varkappa_2)\,\nu_1+(e_1\varkappa+\varkappa_1)\,N; \qquad (3.7)$$

$$2N+N^2+\nu_1^2+\nu_2^2=0,$$
$$\varkappa\,(1+N)+\varkappa_1\nu_1+\varkappa_2\nu_2=\sqrt{\frac{a_1}{Mgl}}\,k_{OZ*}\equiv k,$$
$$d'\varkappa^2+\varkappa_1^2+d_2\varkappa_2^2+2\,(e_1\varkappa_1+e_2\varkappa_2)\,\varkappa-2N=2h'+2 \qquad \left(h'=\frac{h}{Mgl}\right). \qquad (3.8)$$

System (3.7), which describes the general case of oscillations of a heavy solid body about the lower equilibrium position, contains four dimensionless parameters d', d_2, e_1, and e_2, given by formulas (3.6), (2.3), (2.2), and (2.1) (see also Subsection 1.1). The eigenvalues of the matrix of the linear part of system (3.7) are

$$0,\ 0,\ -i,\ +i,\ -\sqrt{d_2}\,i,\ +\sqrt{d_2}\,i, \qquad (3.9)$$

where the simple elementary divisors correspond to a zero eigenvalue. The matrix **S** that has as its elements the correspondingly arranged eigenvectors of the matrix of the linear part of system (3.7) and reduces this system to the Jordan form (in this case diagonal form) and its inverse $\mathbf{S}^{-1}$ are

$$\mathbf{S}=\left\|\begin{matrix} 0 & 1 & 0 & 0 & 0 & 0\\ 0 & -e_1 & 1 & 1 & 0 & 0\\ 0 & -\frac{e_2}{d_2} & 0 & 0 & 1 & 1\\ 1 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & -i\sqrt{d_2} & i\sqrt{d_2}\\ 0 & 0 & i & -i & 0 & 0 \end{matrix}\right\|,$$

$$\mathbf{S}^{-1}=\left\|\begin{array}{cccccc} 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2}e_1 & \frac{1}{2} & 0 & 0 & 0 & -\frac{1}{2}i \\ \frac{1}{2}e_1 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2}i \\ \frac{1}{2}\frac{e_2}{d_2} & 0 & \frac{1}{2} & 0 & \frac{i}{2\sqrt{d_2}} & 0 \\ \frac{1}{2}\frac{e_2}{d_2} & 0 & \frac{1}{2} & 0 & -\frac{i}{2\sqrt{d_2}} & 0 \end{array}\right\|. \tag{3.10}$$

The transition to the Jordan variables by means of the matrix $\mathbf{S}$ reduces system (3.7) to the diagonal form (referring only to the linear part of the reduced system). The reduction to diagonal form precedes the transformation to normal form. The diagonal form *per se* will not be needed in the analysis to follow, but the matrices $\mathbf{S}$ and $\mathbf{S}^{-1}$ will be useful for transforming the system to the Lyapunov form.

2.4. Reduction to the Lyapunov form. The matrix $\mathbf{T}$ of the linear transformation that reduces the system in diagonal form to the skew-symmetric Lyapunov form is

$$\mathbf{T}=\mathbf{I}_2\dot{+}\mathbf{T}_2\dot{+}\mathbf{T}_2,\quad \mathbf{I}_2=\left\|\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right\|,\quad \mathbf{T}_2=\frac{1}{2}\left\|\begin{array}{cc} 1 & -i \\ 1 & i \end{array}\right\|$$

The matrix $\mathbf{L}$ of the resulting transformation of the initial system (3.7) into a Lyapunov-type system is

$$\mathbf{L}=\mathbf{ST},$$

where $\mathbf{S}$ is found from (3.10). The matrix and corresponding inverse are

$$\mathbf{L}=\left\|\begin{array}{cccccc} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -e_1 & 1 & 0 & 0 & 0 \\ 0 & -\frac{e_2}{d_2} & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\sqrt{d_2} \\ 0 & 0 & 0 & 1 & 0 & 0 \end{array}\right\|,$$

$$\mathbf{L}^{-1}=\begin{Vmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ e_1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \frac{e_2}{d_2} & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{\sqrt{d_2}} & 0 \end{Vmatrix}. \tag{4.1}$$

We denote by $\mathbf{u}$ a vector with the components $\varkappa, \varkappa_1, \varkappa_2, \mathrm{N}, \nu_1$, and ν_2 and by $\mathbf{v}$ a vector with the components $v_1, \ldots, v_6$. We write system (3.7) in the form

$$\frac{d\mathbf{u}}{d\tau}=\mathbf{A}\mathbf{u}+\mathbf{g}(\mathbf{u}),$$

where $\mathbf{A}$ is the matrix of its linear part and $\mathbf{g}(\mathbf{u})$ is a vector-function composed of the nonlinear terms of system (3.7). The substitution

$$\mathbf{u}=\mathbf{L}\mathbf{v} \tag{4.2}$$

reduces system (3.7) to the Lyapunov form

$$\frac{d\mathbf{v}}{d\tau}=\mathbf{L}^{-1}\mathbf{A}\mathbf{L}\mathbf{v}+\mathbf{L}^{-1}\mathbf{g}(\mathbf{L}\mathbf{v}). \tag{4.3}$$

In full form, transformation (4.2) and system (4.3) are

$$\begin{gathered}\varkappa=v_2, \quad \varkappa_1=-e_1v_2+v_3, \quad \varkappa_2=-\frac{e_2}{d_2}v_2+v_5, \\ \mathrm{N}=v_1, \quad \nu_1=-\sqrt{d_2}v_6, \quad \nu_2=v_4, \\ v_1=\mathrm{N}, \quad v_2=\varkappa, \quad v_3=e_1\varkappa+\varkappa_1, \quad v_4=\nu_2, \\ v_5=\frac{e_2}{d_2}\varkappa+\varkappa_2, \quad v_6=-\frac{1}{\sqrt{d_2}}\nu_1;\end{gathered} \tag{4.2a}$$

$$\frac{dv_1}{d\tau}=-v_3v_4-d_2\sqrt{d_2}\,v_5v_6,$$

$$\frac{dv_2}{d\tau}=\frac{e_2}{d_2}v_2v_3-e_1d_2v_2v_5+(d_2-1)\,v_3v_5,$$

$$\begin{aligned}\frac{dv_3}{d\tau}=-v_4+\frac{e_2}{d_2}\left(-d'+e_1^2+\frac{e_2^2}{d_2}\right)v_2^2+e_2v_5^2 \\ +\left(d'-d_2-e_1^2-d_2e_1^2-2\frac{e_2^2}{d_2}\right)v_2v_5+e_1d_2v_3v_5,\end{aligned}$$

$$\frac{dv_4}{d\tau}=v_3+v_1v_3+\sqrt{d_2}\left(d'-e_1^2-\frac{e_2^2}{d_2}\right)v_2v_6+\sqrt{d_2}\,e_1v_3v_6+\sqrt{d_2}\,e_2v_5v_6,$$

$$\frac{dv_5}{d\tau}=-\sqrt{d_2}\,v_6+e_1\left(d'-e_1^2-\frac{e_2^2}{d_2}\right)v_2^2-e_1v_3^2$$
$$+\left(1-d'+2e_1^2+\frac{e_2^2}{d_2}+\frac{e_2^2}{d_2^2}\right)v_2v_3+v_2v_5-\frac{e_2}{d_2}v_3v_5,$$
$$\frac{dv_6}{d\tau}=\sqrt{d_2}\,v_5+\sqrt{d_2}\,v_1v_5$$
$$+\frac{1}{\sqrt{d_2}}\left(-d'+e_1^2+\frac{e_2^2}{d_2}\right)v_2v_4-\frac{e_1}{\sqrt{d_2}}v_3v_4-\frac{e_2}{\sqrt{d_2}}v_4v_5. \qquad (4.3a)$$

The first two integrals of (3.8) (the trivial one and that of the kinetic momentum relative to the vertical axis), expressed in new variables, are

$$2v_1+v_1^2+v_4^2+d_2v_6^2=0, \qquad (4.4)$$
$$v_2+v_1v_2-\frac{e_2}{d_2}v_2v_4+e_1\sqrt{d_2}\,v_2v_6-\sqrt{d_2}\,v_3v_6+v_4v_5=k. \qquad (4.5)$$

A linear combination of the energy integral and the trivial integral results in the definite-positive integral

$$v_1^2+\left(d'-e_1^2-\frac{e_2^2}{d_2}\right)v_2^2+v_3^2+v_4^2+d_2(v_5^2+v_6^2)=2h'+2. \qquad (4.6)$$

2.5. Resonances. The ratio of frequencies of the linear part of system (3.7) or (4.3a) is obtained in a straightforward manner from (3.9). Equating this ratio to m/n, where m and n are mutually prime numbers, we obtain

$$d_2=\frac{a_2}{a_1}=\frac{m^2}{n^2}. \qquad (5.1)$$

Formula (2.2) yields

$$\sin^2\varepsilon_X=\frac{1}{2\mathcal{R}}[\mathcal{R}+(J_{XX}J_{YY}-J_{XY}^2)-(J_{XX}J_{ZZ}-J_{ZX}^2)],$$
$$\cos^2\varepsilon_X=\frac{1}{2\mathcal{R}}[\mathcal{R}-(J_{XX}J_{YY}-J_{XY}^2)+(J_{XX}J_{ZZ}-J_{ZX}^2)],$$
$$\sin 2\varepsilon_X=\frac{2J_{XX}J_{YY}}{\mathcal{R}},$$

where

$$\mathcal{R}=+\sqrt{[J_{XX}(J_{ZZ}-J_{YY})+J_{XY}^2-J_{ZX}^2]^2+4J_{XY}^2J_{ZX}^2}.$$

By formulas (2.3) and (2.1). we derive

$$d_2=\frac{a_2}{a_1}=\frac{J_{XX}(J_{YY}+J_{ZZ})-(J_{XY}^2+J_{ZX}^2)-\mathcal{R}}{J_{XX}(J_{YY}+J_{ZZ})-(J_{XY}^2+J_{ZX}^2)+\mathcal{R}}.$$

After elementary manipulations, the resonant equation (5.1) reduces to

$$(n^4+m^4)\,J_{XX}\,(J_{XX}J_{YY}J_{ZZ}-J_{YY}J_{ZX}^2-J_{ZZ}J_{XY}^2)$$
$$=n^2m^2\,[(J_{XX}J_{YY}-J_{XY}^2)^2+(J_{XX}J_{ZZ}-J_{ZX}^2)^2+2J_{XY}^2J_{ZX}^2]. \quad (5.2)$$

We assume that the base axes are the principal axes of the ellipsoid of inertia for the point O, that is, $J_{XY}=J_{ZX}=0$. In this case these axes are also the special ones, because formula (2.2) yields $\varepsilon_X=0$. As a result, equation (5.2) becomes

$$(n^4+m^4)\,J_{YY}J_{ZZ}=n^2m^2\,(J_{YY}^2+J_{ZZ}^2). \quad (5.3)$$

Obviously, its solution is

$$\frac{J_{ZZ}}{J_{YY}}=\frac{m^2}{n^2}. \quad (5.4)$$

For instance, in the Lagrange-Poisson case we obtain $J_{XX}=C$ (because axis OX is directed to the centroid) and $J_{YY}=J_{ZZ}=A$, which coincides with (5.4) for $m=n=1$.

In the Kovalevskaya case [92], however, $J_{XX}=A$ and

$$\frac{J_{ZZ}}{J_{YY}}=\frac{1}{2},$$

that is, (5.4) is not satisfied.

2.6. Application of the method of successive approximations. We solve integrals (4.4) and (4.5) with respect to the variables v_1 and v_2

$$v_1=-1+\sqrt{1-v_4^2-d_2v_6^2}=-\frac{1}{2}v_4^2-\frac{1}{2}d_2v_6^2+(4), \quad (6.1)$$

$$v_2=\frac{k+\sqrt{d_2}\,v_3v_6-v_4v_5}{1+v_1-\frac{e_2}{d_2}v_4+e_1\sqrt{d_2}\,v_6}=k+\frac{e_2}{d_2}kv_4-e_1\sqrt{d_2}\,kv_6+(3). \quad (6.2)$$

If

$$v_1^2+v_2^2<1, \quad -1<N \quad (6.3)$$

and if (4.2a) is taken into account, then expansions (6.1) and (6.2) are convergent.

These conditions are obviously satisfied if the centroid of the oscillating body does not reach the horizontal plane passing through the fixed point O.

The last four equations of (4.3a) can be presented in vector form as

$$\frac{d\mathbf{w}}{d\tau}=\mathbf{B}\mathbf{w}+\mathbf{f}\,(\mathbf{w},\,v_1(\mathbf{w}),\,v_2(\mathbf{w})),$$

$$\mathbf{w}=\begin{pmatrix}v_3\\ v_4\\ v_5\\ v_6\end{pmatrix}, \quad \mathbf{B}=\begin{Vmatrix}0 & -1 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 0 & 0 & -\sqrt{d_2}\\ 0 & 0 & \sqrt{d_2} & 0\end{Vmatrix}, \quad (6.4)$$

where $\mathbf{f}$ is a vector-function composed of the nonlinear terms. As the zeroth approximation of system (6.4) we choose a vector composed of the initial conditions $\mathbf{w}_0 = \mathbf{w}(0)$, and as the first approximation a solution of the system

$$\frac{d\mathbf{w}_1}{d\tau} = \mathbf{B}\mathbf{w}_1 + \mathbf{f}(\mathbf{w}_0, v_1(\mathbf{w}_0), v_2(\mathbf{w}_0)).$$

With the same initial conditions $\mathbf{w}_1(0) = \mathbf{w}_0$, this solution becomes

$$\mathbf{w}_1(\tau) = e^{\tau\mathbf{B}}\mathbf{w}_0 + \mathbf{B}^{-1}(e^{\tau\mathbf{B}} - \mathbf{I}_4)\,\mathbf{f}(\mathbf{w}_0, v_1(\mathbf{w}_0), v_2(\mathbf{w}_0)).$$

The expressions for the components of $\mathbf{w}_1(\tau)$ are

$$\begin{aligned}
v_3^1(\tau) &= v_3(0)\cos\tau - v_4(0)\sin\tau \\
&\quad + \Big[\frac{e_2}{d_2}\Big(-d' + e_1^2 + \frac{e_2^2}{d_2}\Big)v_2^2(0) + e_2 v_5^2(0) \\
&\quad + \Big(d' - d_2 - e_1^2 - d_2 e_1^2 - 2\frac{e_2^2}{d_2}\Big)v_2(0)v_5(0) + e_1 d_2 v_3(0)v_5(0)\Big]\sin\tau \\
&\quad - 2\Big[v_1(0)v_3(0) + \sqrt{d_2}\Big(d' - e_1^2 - \frac{e_2^2}{d_2}\Big)v_2(0)v_6(0) \\
&\quad + \sqrt{d_2}\,e_1 v_3(0)v_6(0) + \sqrt{d_2}\,e_2 v_5(0)v_6(0)\Big]\sin^2\frac{\tau}{2}, \\
v_4^1(\tau) &= v_3(0)\sin\tau + v_4(0)\cos\tau \\
&\quad + 2\Big[\frac{e_2}{d_2}\Big(-d' + e_1^2 + \frac{e_2^2}{d_2}\Big)v_2^2(0) + e_2 v_5^2(0) \\
&\quad + \Big(d' - d_2 - e_1^2 - d_2 e_1^2 - 2\frac{e_2^2}{d_2}\Big)v_2(0)v_5(0) \\
&\quad + e_1 d_2 v_3(0)v_5(0)\Big]\sin^2\frac{\tau}{2} \\
&\quad + \Big[v_1(0)v_3(0) + \sqrt{d_2}\Big(d' - e_1^2 - \frac{e_2^2}{d_2}\Big)v_2(0)v_6(0) \\
&\quad + \sqrt{d_2}\,e_1 v_3(0)v_6(0) + \sqrt{d_2}\,e_2 v_5(0)v_6(0)\Big]\sin\tau, \\
v_5^1(\tau) &= v_5(0)\cos\sqrt{d_2}\,\tau - \sqrt{d_2}\,v_6(0)\sin\sqrt{d_2}\,\tau \\
&\quad + \frac{1}{\sqrt{d_2}}\Big[e_1\Big(d' - e_1^2 - \frac{e_2^2}{d_2}\Big)v_2^2(0) - e_1 v_3^2(0) \\
&\quad + \Big(1 - d' + 2e_1^2 + \frac{e_2^2}{d_2} + \frac{e_2^2}{d_2^2}\Big)v_2(0)v_3(0) \\
&\quad + v_2(0)v_5(0) - \frac{e_2}{d_2}v_3(0)v_5(0)\Big]\sin\sqrt{d_2}\,\tau \\
&\quad - 2\Big[v_1(0)v_5(0) + \frac{1}{d_2}\Big(-d' + e_1^2 + \frac{e_2^2}{d_2}\Big)v_2(0)v_4(0) \\
&\quad - \frac{e_1}{d_2}v_3(0)v_4(0) - \frac{e_2}{d_2}v_4(0)v_5(0)\Big]\sin^2\frac{\sqrt{d_2}\,\tau}{2},
\end{aligned}$$

$$
\begin{aligned}
v_6^1(\tau) = {} & \sqrt{d_2}\, v_5(0)\sin\sqrt{d_2}\,\tau + v_6(0)\cos\sqrt{a_2}\,\tau \\
& + \frac{2}{\sqrt{d_2}}\Big[e_1\Big(d' - e_1^2 - \frac{e_2^2}{d_2}\Big)v_2^2(0) - e_1 v_3^2(0) \\
& + \Big(1 - d' + 2e_1^2 + \frac{e_2^2}{d_2} + \frac{e_2^2}{d_2^2}\Big)v_2(0)\,v_3(0) \\
& + v_2(0)\,v_5(0) - \frac{e_2}{d_2}v_3(0)\,v_5(0)\Big]\sin^2\frac{\sqrt{d_2}\,\tau}{2} \\
& + \Big[v_1(0)\,v_5(0) + \frac{1}{d_2}\Big(-d' + e_1^2 + \frac{e_2^2}{d_2}\Big)v_2(0)\,v_4(0) \\
& \qquad - \frac{e_1}{d_2}v_3(0)\,v_4(0) - \frac{e_2}{d_2}v_4(0)\,v_5(0)\Big]\sin\sqrt{d_2}\,\tau.
\end{aligned}
$$

Now $v_1^1(\tau)$ and $v_2^1(\tau)$ can be found from (6.1) and (6.2). The second approximation $\mathbf{w}_2(\tau)$ is then found from the equation

$$\frac{d\mathbf{w}_2}{d\tau} = \mathbf{B}\mathbf{w}_2 + \mathbf{f}(\mathbf{w}_1(\tau),\ v_1^1(\tau),\ v_2^1(\tau)),$$

whose solution for $\mathbf{w}_2(0) = \mathbf{w}_0$ yields

$$\mathbf{w}_2(\tau) = e^{\tau\mathbf{B}}\mathbf{w}_0 + e^{\tau\mathbf{B}}\int_0^\tau e^{-s\mathbf{B}}\mathbf{f}(\mathbf{w}_1(s),\ v_1^1(s),\ v_2^1(s))\,ds. \tag{6.5}$$

This result can be somewhat weakened by setting $v_1^1(s) = 0$ and $v_2^1(s) = k$ in (6.5). This means that the terms of the third order of smallness in the expression for $\mathbf{f}$ are neglected in (6.5).

By formulas (4.2a), integral (4.6) can be recast in the initial variables

$$
\begin{aligned}
\Big(d' - e_1^2 - \frac{e_2^2}{d_2}\Big)\varkappa^2 + (e_1\varkappa + \varkappa_1)^2 + d_2\Big(\frac{e_2}{d_2}\varkappa + \varkappa_2\Big)^2 + \mathrm{N}^2 + \nu_1^2 + \nu_2^2 & \\
= 2h' + 2. & \qquad (6.6)
\end{aligned}
$$

We assume that the centroid of the body has reached the horizontal plane passing through the fixed point O ($\mathrm{N} = -1$, $\nu_1^2 + \nu_2^2 = 1$) with the zero angular velocity ($\varkappa = \varkappa_1 = \varkappa_2 = 0$). Substituting this into (6.6), we find $h' = 0$. It was mentioned that this situation is a limiting one for the convergence of the suggested version of the method of successive approximations. Therefore, taking into account that $h' = 0$, we see that integral (4.6) defines the surface of a six-dimensional ellipsoid containing the allowed initial values of the variables $v_1, \ldots, v_6$; in other words, the inequality

$$v_1^2(0) + \Big(d' - e_1^2 - \frac{e_2^2}{d_2}\Big)v_2^2(0) + v_3^2(0) + v_4^2(0) + d_2 v_5^2(0) + d_2 v_6^2(0) < 2$$

must be satisfied.

BRIEF BIBLIOGRAPHICAL NOTES

Part One

Chapter I

§ 1. A practical method of finding periodic solutions of Lyapunov systems was suggested by Malkin [111a]. Sokolov [369] determined periodic solutions of certain types of Lyapunov systems for which the available methods of calculation proved unmanageable.

1.1. These results are taken from [371e-g, j, s].

1.2. This transformation was given in [371j, n, s, t].

§ 2. Here we follow § IV.6 of [80].

The Poincaré method for both nonautonomous and autonomous systems can be found in monographs by Andronov, Vitt, Khaikin [5], Blekhman [20a], Bulgakov [31], MacMillan [109], and Malkin [111a, b]. Periodic solutions by the method of small parameter were practically investigated by Proskuryakov [348a-n] and Plotnikova [341a-c]. These papers treat quasilinear autonomous and nonautonomous systems with one or several degrees of freedom, certain nonlinear systems, and some special cases.

Let us discuss one nontraditional application of the Poincaré method: the study of the motion of a heavy solid body about a fixed point, including both an analysis of the general properties of the equations of motion and the actual integration. In 1953 Sretenskii [370] suggested applying the method of small parameter to analyze the motion of a solid body rotating at a high angular velocity about a fixed point. Arkhangelskii [216a, c] discussed a method of constructing periodic solutions of quasilinear autonomous systems that possess first integrals; this results in the appearance of degenerate situations. Later these results were used [216b, d, e] to derive new special solutions for a heavy solid body rotating at a high angular velocity about a fixed point.

Chapter II

Simple, physical, and elastic compound pendulums are treated extensively in the literature. Papers by Mettler [320a], Struble [376b], Heinbockel and Struble [269, 379a, b], and Cheshankov [244a-e] contain an extensive bibliography.

§ 1. The results are taken from [371d].

1.6. Litvin-Sedoi [304a, b] demonstrated that if a simple plane pendulum of variable length is mounted on a suspension undergoing a nonzero acceleration, a restoring force due to the inertial force of the rotational acceleration of the pendulum is produced even in the absence of external forces.

§ 2. The results are taken from [371c].

Chapter III

The method of averaging is given in monographs by Bogolyubov ([22], vol. I), Mitropolskii [127d], Volosov and Morgunov [204], and Grebenikov and Ryabov ([66a], Pt. I).

The problems of accuracy in the theory of nonlinear oscillations were discussed by Ryabov [355b-e] and Gorbatenko [260a-d].

The results presented are taken from [371h] (§ 1), [371k] (§ 2), [371n] (§ 3), [372] (§ 4).

§ 1. The first results dealing with energy transfer in oscillations of elastic pendulums were obtained by Vitt and Gorelik [394]. Energy transfer in flexural-torsional oscillations was analyzed by Kononenko [290a]. Energy transfer was also analyzed by Struble [376c], Struble and Heinbockel [379a, b], Struble and Warmbrod [380], Cheshankov [244a, b], Zelman [398a], and Mercer, Rees, and Fahy [319].

Chapter IV

§ 1. Lyapunov systems were investigated by Malkin [111a, b]. Chapter VIII of [111b] discusses nearly Lyapunov systems, but in a different sense than here, namely, nonautonomous systems.

1.1, 1.2, 1.5. The results are taken from [371m].

1.2. A proof of the Poincaré theorem on the expansion of integrals of analytic systems of differential equations in powers of a small parameter can also be found in a monograph by Golubev ([65a], Ch. III, § 2).

1.3. The introduction to Subsection 1.3 shows that the discussion goes beyond the scope of the equation $\ddot{x}+\omega^2 x=\varepsilon\varphi(x, \dot{x})$. This equation is extensively treated in the literature both in the scalar and in the vector case; among the publications in the USSR we mention the monographs by Andronov, Vitt, and Khaikin [5], Bulgakov [31], Malkin [111a, b], Bogolyubov and Mitropolskii [23a], Moiseev [129], and Roitenberg [166]. It should be emphasized at the same time that Subsection 1.3 is a supplementary analysis of the general case of oscillations in a system with one degree of freedom described by the equation $\ddot{x}+f(x)=\varepsilon\varphi(x)$, as given by Bulgakov [31], Moiseev [129], and Roitenberg [166].

1.3, 1.4. The results are taken from [371y].

§ 2. Nearly Lyapunov systems were investigated by Malkin [111b], who analyzed periodic solutions that reduce to the Lyapunov solutions for $\mu=0$ and periodic solutions in the resonant case. The problems of existence, stability, and practical calculation of periodic solutions were studied by Shimanov [364a, c, d] by a method of auxiliary systems that he suggested.

One special case of Lyapunov-type systems was analyzed by Sokolov [369]; Ryabov used a somewhat different approach to similar systems [355a].

Part Two

The reader should be warned about terminology. Here we consider normal forms of systems of differential equations and their applications to oscillation problems.

The theory of oscillations also considers normal mode oscillations, which have rectilinear trajectories in the configurational space. Let us briefly describe the literature. The definition of normal mode oscillations can be found in [351a, b]. The existence of solutions close to normal mode oscillations was proved by Lyapunov [108a] for a wide class of quasilinear systems. Manevich [308] demonstrated the possibility of interpreting normal mode oscillations in terms of the theory of group-invariant solutions developed by Ovsyannikov [142]. Manevich and Pinskii [311a, b] found some new classes of systems allowing both rectilinear and

curvilinear normal modes of oscillations, since the corresponding systems are invariant with respect to the complementary discrete groups of the transformations. Manevich and Cherevatskii [309a, b] studied degenerate systems that can be reduced to singularly perturbed equations. An efficient method was suggested for constructing periodic solutions of such systems. A constructive method of finding normal mode oscillations of multidimensional conservative systems was given in [310] and by Mikhlin in [321a]. The method is based on constructing curvilinear trajectories of the solutions to be found, tracing them in the configurational space in the vicinity of the rectilinear forms of oscillations. The trajectories are constructed as power series, and the relationship to Lyapunov's results [108a] is established. Rosenberg [351a] proved some theorems on the existence of normal mode oscillations. Mikhlin [321b] investigated resonant modes and proved that in several special cases they are close to normal mode oscillations.

Chapter V

Elementary information on autonomous systems can be found in a monograph by Myškis [135]. For advanced reading we recommend a monograph by Daletskii and Krein [47], which a nonmathematician can reinterpret in terms of finite-dimensional space. A qualitative theory is given in a monograph by Nemytskii and Stepanov [139], although normal forms are treated only in the first edition (1947).

§ 1. The fundamental results were obtained by Brjuno [238j]; this reference also contains the relevant bibliography.

1.2. The fundamental Brjuno theorem [238j] does not assume that the Jordan form is diagonal, that is, it treats the case of arbitrary elementary divisors of the linear part of the system. The presentation in this subsection is based upon the statement of the problem in 1.1.

1.3. The Poincaré theorem (in the case of violation of conditions (2) or (1)) was generalized by Siegel [183] and Pliss [340].

§ 2. A geometrical interpretation of "truncated systems" (in the space of exponents) and examples of the use of these systems can be found in [238m].

2.4. The existence of an analytic transformation in the real case was discussed by Markhashov [315] and Brjuno [238v]. Poincaré theorem was generalized (in the sense of the existence of a piecewise-smooth transformation of the linear system) by Samovol [357] and Brjuno [238v].

§ 3. The author agrees with Brjuno's remark [238l] that the method of normal forms is naturally separated into two parts. The first (algebraic) part consists in indicating an algorithm for constructing the necessary formal expansions [238j]. In this book the algorithms are elaborated to the point of recurrent calculation formulas covering the general case.

The second (analytic) part consists in interpreting the results by means of analytic functions (Poincaré [149a], Dulac [253], Siegel [183], Pliss [340], and, finally, Brjuno [238j]) or smooth functions (Birkhoff [15], and others) and evaluating the accuracy of approximate integration by this method (Birkhoff [15], Siegel [183], and Moser [132]).

The local method is not restricted to normal forms. For the further study of this method (seminormal forms and integral manifolds related to them, etc.) we refer the reader to Brjuno's works [238j, p-u], which also contain the relevant references.

The extension of normal forms to nonautonomous systems is outlined in a paper by Kostin [294].

Chapter VI

The results are taken from [371i].

§ 1. The first (after Poincaré [149a]) to express the general solution of equations (V, 1, 1.1) in terms of the solution of equations (V, 1, 3.2) was Lyapunov [108a].

The relationship of the problems covered in this section to the first Lyapunov method, as well as a number of more general topics, will be found by the reader in Part II of a review by Erugin [54b].

1.2. The general solution in which arbitrary constants are chosen as the initial values of variables (and of their derivatives) will be referred to as a solution of the Cauchy problem in the general case.

Chapter VII

§ 1. The results are taken from [371p].

1.4. The problem was formulated by Roitenberg. The effects of play and friction on the free and forced oscillations are discussed in [371a].

§ 2. The results are taken from [371u].

Chapter VIII

1.4. A more general approach to the investigation of critical cases of stability (for example, with right-hand sides not necessarily analytic) is outlined in a monograph by Krasovskii [100].

The transformations of this subsection are a special case of power series developed by Brjuno [238c].

1.5. The contents of the Bibikov-Pliss theorem [227] are intentionally curtailed by the author.

2.1, 2.4-2.7. The results are taken from [371l].

§ 4. The section presents the contents of [187b, Pt. II, § 6].

An analysis of the critical case of three pairs of pure imaginary eigenvalues of the matrix of the linear part can be found in a paper by Veretennikov [392b].

Chapter IX

Recent investigations of Kharlamova and Kharlamov and their school on the dynamics of a solid body and of systems of bodies are collected in [119]. An extensive bibliography of the most recent works (seventy references) is cited in a paper by Kharlamov [280].

Integral manifolds and bifurcation sets used in the treatment of the motion of a solid body with a fixed point are described by Katok [277] on the basis of the topology of mechanical systems with symmetry, as introduced by Smale [368].

1.4. Concerning problems of the stability of permanent rotations the reader is referred to Rumyantsev's work (see bibliography in [171]).

1.7. Another version of the method of successive approximations is given in [371w].

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